Assignment 3.1

a) Plain strain matrix can be written in the form of

$$\frac{E^*}{1-\nu^{*2}}\begin{bmatrix}1 & \nu^* & \\ \nu^* & 1 & \\ & & \frac{1-\nu^*}{2}\end{bmatrix} \text{ (a) it is equivelant to } \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \\ \nu & 1-\nu & \\ & & \frac{1-2\nu}{2}\end{bmatrix} \text{ (b)}$$

$$\Rightarrow \begin{cases}\frac{E^*}{1-\nu^{*2}} = \frac{E}{(1+\nu)(1-2\nu)}(1-\nu) \quad ① \\ \frac{E^*}{1-\nu^{*2}}\nu^* = \frac{E}{(1+\nu)(1-2\nu)}\nu \quad ②\end{cases} \quad ②/① \Rightarrow \nu^* = \frac{\nu}{1-\nu} \quad ③$$

Substitute ③ into ①

$$\Rightarrow E^* = \frac{E}{1-\nu^2}$$

Check the last term in the matrix (a)

$$\frac{E^*}{1-\nu^{*2}}\frac{1-\nu^*}{2} = \frac{1}{2}\frac{E^*}{1+\nu^*} = \frac{E}{1-\nu^2}\frac{1}{2}(1-\nu) = \frac{1}{2}\frac{E}{1+\nu}$$ the same as the last term in matrix (b)

Therefore $\begin{cases}E^* = \frac{E}{1-\nu^2} \\ \nu^* = \frac{\nu}{1-\nu}\end{cases}$

b) Inverse process

Plain stress matrix can be written as

$$\frac{E^*}{(1+\nu^*)(1-2\nu^*)}\begin{bmatrix}1-\nu^* & \nu^* & \\ \nu^* & 1-\nu^* & \\ & & \frac{1-2\nu^*}{2}\end{bmatrix} \text{ (c) it is equivelant to } \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & \\ \nu & 1 & \\ & & \frac{1-\nu}{2}\end{bmatrix} \text{ (d)}$$

$$\Rightarrow \begin{cases}\frac{E^*}{(1+\nu^*)(1-2\nu^*)}(1-\nu^*) = \frac{E}{(1-\nu^2)} \quad ⑤ \\ \frac{E^*}{(1+\nu^*)(1-2\nu^*)}\nu^* = \frac{E}{(1-\nu^2)}\nu \quad ⑥\end{cases} \quad ⑤/⑥ \Rightarrow \frac{1}{\nu} = \frac{1-\nu^*}{\nu^*} \Rightarrow \nu^* = \frac{\nu}{1+\nu} \quad ⑦$$

Subsitute ⑦ into ⑥

$$\Rightarrow E^* = \left(1+\frac{\nu}{1+\nu}\right)\left(1-2\frac{\nu}{1+\nu}\right)\frac{(1+\nu)}{\nu}\frac{E}{(1-\nu^2)}\nu$$

$$= E\frac{1+2\nu}{(1+\nu)^2}$$

Check the last term in the matrix (c)

$$\frac{E^*}{(1+\nu^*)(1-2\nu^*)}\frac{1-2\nu^*}{2} = E\frac{1+2\nu}{(1+\nu)^2}\frac{1+\nu}{1+2\nu}\frac{1+\nu}{1-\nu}\frac{1}{2}\frac{1-\nu}{1+\nu} = \frac{E}{2(1+\nu)}$$

It is the same as the last term in matrix (d)

Therefore $\begin{cases} E^* = E\dfrac{1+2\nu}{(1+\nu)^2} \\ \nu^* = \dfrac{\nu}{1+\nu} \end{cases}$

Assignment 3.2

a) $\begin{cases} \lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)} & ① \\ \mu = \dfrac{E}{2(1+\nu)} & ② \end{cases}$

① devided by ②

$$\Rightarrow \frac{\lambda}{\mu} = \frac{2\nu}{1-2\nu}$$

$$\Rightarrow 1+\frac{\lambda}{\mu} = \frac{1}{1-2\nu}$$

$$\Rightarrow \frac{1}{1+\lambda/\mu} = 1-2\nu$$

$$\Rightarrow \nu = \frac{1}{2}\left[1-\frac{1}{1+\lambda/\mu}\right] = \frac{1}{2}\frac{\lambda}{\lambda+\mu} \qquad ③$$

Substitute ③ into ②

$$\Rightarrow E = 2\mu\left(1+\frac{1}{2}\frac{\lambda}{\lambda+\mu}\right) = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$

Therefore, the inverse relation is

$$\begin{cases} E = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu} \\ \nu = \dfrac{\lambda}{2(\lambda+\mu)} \end{cases}$$

b) • Elastic matrix for plane stress is

$$\frac{E}{(1+\nu)(1-\nu)}\begin{bmatrix} 1 & \nu & \\ \nu & 1 & \\ & & \frac{1-\nu}{2} \end{bmatrix}$$

$$1+\nu = \frac{3\lambda+2\mu}{2\lambda+2\mu}$$

$$1-\nu = \frac{\lambda+2\mu}{2\lambda+2\mu}$$

The matrix becomes

$$= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\,\frac{2\lambda+2\mu}{3\lambda+2\mu}\,\frac{2\lambda+2\mu}{\lambda+2\mu}\begin{bmatrix} 1 & \frac{\lambda}{2\lambda+2\mu} & \\ \frac{\lambda}{2\lambda+2\mu} & 1 & \\ & & \frac{\lambda+2\mu}{4\lambda+4\mu} \end{bmatrix}$$

$$= \frac{2\mu}{\lambda+2\mu}\begin{bmatrix} 2\lambda+2\mu & \lambda & \\ \lambda & 2\lambda+2\mu & \\ & & \frac{\lambda+2\mu}{2} \end{bmatrix}$$

- Plain strain elastic matrix

$$\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-2\nu & \nu & \\ \nu & 1-\nu & \\ & & \frac{1-2\nu}{2} \end{bmatrix}$$

$$1-2\nu = 1-\frac{\lambda}{\lambda+\mu} = \frac{\mu}{\lambda+\mu}$$

The matix becomes

$$\frac{\mu(3\lambda+2\mu)}{\cancel{\lambda+\mu}}\frac{(2\lambda+2\mu)}{3\lambda+2\mu}\frac{\cancel{\lambda+\mu}}{\mu}\begin{bmatrix} \frac{\lambda+2\mu}{2\lambda+2\mu} & \frac{\lambda}{2(\lambda+\mu)} & \\ \frac{\lambda}{2(\lambda+\mu)} & \frac{\lambda+2\mu}{2(\lambda+\mu)} & \\ & & \frac{\mu}{2(\lambda+\mu)} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda+2\mu & \lambda & \\ \lambda & \lambda+2\mu & \\ & & \mu \end{bmatrix}$$

C) The stress-strain matrix of plan strain

$$\underset{\sim}{E} = \begin{bmatrix} \lambda+2\mu & \lambda & \\ \lambda & \lambda+2\mu & \\ & & \mu \end{bmatrix} = \begin{bmatrix} \lambda & \lambda & \\ \lambda & \lambda & \\ & & 0 \end{bmatrix} + \begin{bmatrix} 2\mu & & \\ & 2\mu & \\ & & \mu \end{bmatrix} = \underset{\sim}{E}_\lambda + \underset{\sim}{E}_\mu$$

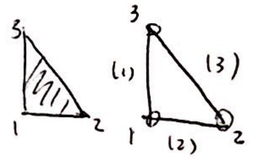
where $\underset{\sim}{E}_\lambda = \lambda\begin{bmatrix} 1 & 1 & \\ 1 & 1 & \\ & & 0 \end{bmatrix}$ $\underset{\sim}{E}_\mu = \mu\begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix}$

d) Express $\underset{\sim}{E}_\lambda, \underset{\sim}{E}_\mu$ interms of E, ν

$$\underset{\sim}{E}_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & 1 & \\ 1 & 1 & \\ & & 0 \end{bmatrix}$$

$$\underset{\sim}{E}_\mu = \frac{E}{2(1+\nu)}\begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix}$$

Assignment 3.3



$a = h = 1$

E, ν.

$A_1 = A_2, A_3$

a) Calculate $\underset{\sim}{K}^e$

- Plane linea triangle. Plane stress

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \xi_1 & 0 & \xi_2 & 0 & \xi_3 & 0 \\ 0 & \xi_1 & 0 & \xi_2 & 0 & \xi_3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \underset{\sim}{N} u^e$$

kinematic equations

$$\underset{\sim}{\varepsilon} = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} u^e = \underset{\sim}{B} u^e$$

Plane stress constitutive

$$\underset{\sim}{\sigma} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & \\ \nu & 1 & \\ & & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{bmatrix} = \underset{\sim}{E} \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{K}^e = \int_{\Omega^e} h \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \, d\Omega$$

$$= \frac{h}{4A^2} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \underset{\sim}{E} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \int_{\Omega^e} d\Omega$$

$$= \frac{h}{4A} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & \\ \nu & 1 & \\ & & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

For $\nu = 0$, $h = 1$

$$\underset{\sim}{K}^e_{tri} = \frac{1}{4 \times \frac{1}{2}} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} E \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{E}{2} \begin{bmatrix} 1.5 & 0.5 & -1 & -0.5 & -0.5 & 0 \\ 0.5 & 1.5 & 0 & -0.5 & -0.5 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

For $\nu = 0$, the constituitive laws are the same for the plane stress and plain strain. So the stiffiness matrices are the same.

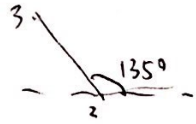
$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \\ \frac{\nu}{1-\nu} & 1 & \\ & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} = E \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{2} \end{bmatrix}$$

- Bars

$$\underset{\sim}{k}^{①} = \frac{EA_1}{a}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = EA\cdot\begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{c} 1x \\ 1y \\ 3x \\ 3y \end{array}$$

$$\underset{\sim}{k}^{②} = \frac{EA_2}{a}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = EA\cdot\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{array}{c} 1x \\ 1y \\ 2x \\ 2y \end{array}$$

$$\underset{\sim}{k}^{③} = \frac{EA_3}{\sqrt{2}a}\begin{bmatrix} C^2 & SC & -C^2 & -SC \\ SC & S^2 & -SC & -S^2 \\ -C^2 & -SC & C^2 & SC \\ -SC & -S^2 & SC & S^2 \end{bmatrix}$$



$$= \frac{EA_3}{\sqrt{2}}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \frac{EA'}{2\sqrt{2}}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{array}{c} 2x \\ 2y \\ 3x \\ 3y \end{array}$$

Assemble them to obtain the global matix

$$\underset{\sim}{k}^{e}_{bar} = \begin{bmatrix} EA & 0 & -EA & 0 & 0 & 0 \\ & EA & 0 & 0 & 0 & -EA \\ & & EA+\frac{EA'}{2\sqrt{2}} & -\frac{EA'}{2\sqrt{2}} & -\frac{EA'}{2\sqrt{2}} & \frac{EA'}{2\sqrt{2}} \\ & \text{sym} & & \frac{EA'}{2\sqrt{2}} & \frac{EA'}{2\sqrt{2}} & -\frac{EA'}{2\sqrt{2}} \\ & & & & \frac{EA'}{2\sqrt{2}} & -\frac{EA'}{2\sqrt{2}} \\ & & & & & EA+\frac{EA'}{2\sqrt{2}} \end{bmatrix} \quad (2)$$

(b) Compare (1) and (2), we observe that there is no way to make them equivalent as $(\underset{\sim}{k}^{e}_{tri})_{12} \neq (\underset{\sim}{k}^{e}_{bar})_{12}$

To make them as similar as possible. Firstly consider the diagonal term $(\underset{\sim}{k}^{e}_{tri})_{11} = (\underset{\sim}{k}^{e}_{bar})_{11}$.

$\Rightarrow EA = 0.75E \quad \Rightarrow A = 0.75$

Let $(\underset{\sim}{k}^{e}_{tri})_{44} = (\underset{\sim}{k}^{e}_{bar})_{44}$

$\Rightarrow \frac{EA'}{2\sqrt{2}} = 0.25E \Rightarrow A' = \frac{1}{\sqrt{2}}$

(c) The two stiffness matrices can not be equivalent because:

① In the triangular model, there exists shear stress while in the bar systerm, we only have axial stress along the bar.

② In the triangular model, to interpolate the displacement at one point, the displacement at three nodes are taken into consideration

$$\bar{u} = N_i \bar{u}_i$$

However, in the bar model, the displacement of a point on the bar is only influenced by the two nodes of that bar. Moreover, there is no material inside the triangle.

③ Another angle of looking at the problem is from the derivation of stiffness matrix based on MPE. The energy function of the triangular plane system is evaluated at the whole domain while that of the bar system is only evaluated along the bar. Moreover, their constituitive laws are different. Therefore, the stiffness matrices cannot be the same.

(d) When $\nu \neq 0$, the stiffness matrices for plane strain and plane stress are not the same.

Plane stress:

$$k^e_{tri} = \frac{E}{2(1-\nu^2)} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \nu & \\ \nu & 1 & \\ & & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Plane strain

$$k^e_{tri} = \frac{E(1-\nu)}{2(1+\nu)(1-2\nu)} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \\ \frac{\nu}{1-\nu} & 1 & \\ & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

For the plane stress problem, $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ will be non-zero.

If we compare it with the bar system, in the bar system, we can only recover the axial stess along the bar element. And it is also not able to provide any information of transverse strain/stress.