INTERNATIONAL CENTRE FOR
NUMERICAL METHODS IN ENGINEERING

UNIVERSITAT POLITÈCNICA DE CATALUNYA

MASTER OF SCIENCE IN COMPUTATIONAL MECHANICS

Computational Structural Mechanics and Dynamics

Assignment 3

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ASSIGNMENT 3.1

Suppose that the structural material is isotropic, with elastic modulus E and Poisson's ratio ν. The in-plane stress-strain relations for plane stress and plane strain as given in any textbook on elasticity are:

$$plane\ stress: \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix} \tag{0.1}$$

$$plane\ strain: \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix} \tag{0.2}$$

***a)** Show that the constitutive matrix of plane strain can be formally obtained by replacing E by a fictitious modulus E* and ν by a fictitious Poisson's ratio ν^* in the plane stress constitutive matrix. Find the expression of E* and ν^* in terms of E and ν.*

Solution:
Using the next two equations:

$$\epsilon_{xx} = \frac{1-\nu^2}{E}\left[\sigma_{xx} - \left(\frac{\nu}{1-\nu}\right)\sigma_{yy}\right] \tag{0.3}$$

$$\epsilon_{yy} = \frac{1-\nu^2}{E}\left[\sigma_{yy} - \left(\frac{\nu}{1-\nu}\right)\sigma_{xx}\right] \tag{0.4}$$

Then, these equations can be reduced using E^* and ν^* as:

$$\epsilon_{xx} = \frac{1}{E^*}\left[\sigma_{xx} - \nu^*\sigma_{yy}\right] \tag{0.5}$$

$$\epsilon_{yy} = \frac{1}{E^*}\left[\sigma_{yy} - \nu^*\sigma_{xx}\right] \tag{0.6}$$

where:

$$E^* = \frac{E}{1-\nu^2} \tag{0.7}$$

$$\nu^* = \frac{\nu}{1-\nu} \tag{0.8}$$

Now, considering the plane stress matrix of equation 0.1 and the mechanical properties of the equations 0.7 and 0.8 as:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E^*}{1-\nu^{*2}} \begin{bmatrix} 1 & \nu^* & 0 \\ \nu^* & 1 & 0 \\ 0 & 0 & \frac{1-\nu^*}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix}$$

Performing the corresponding substitutions:

$$\frac{E}{(1-\nu^2)\left(1-\left(\frac{\nu}{1-\nu}\right)^2\right)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1}{2}\left(1-\frac{\nu}{1-\nu}\right) \end{bmatrix}$$

$$\frac{E}{(1-\nu)(1+\nu)\left(1-\frac{\nu}{1-\nu}\right)\left(1+\frac{\nu}{1-\nu}\right)(1-\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

$$\frac{E}{(1-\nu)(1+\nu)\left(\frac{1-\nu-\nu}{1-\nu}\right)\left(\frac{1+\nu-\nu}{1-\nu}\right)(1-\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

$$\frac{E(1-\nu)(1-\nu)}{(1-\nu)(1+\nu)(1-2\nu)(1)(1-\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1}{2}\frac{1-2\nu}{(1-\nu)} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix}$$

which corresponds to the equation of the plane strain constitutive matrix.

b) *Do also the inverse process: go from plane strain to plain stress by replacing a fictitious modulus and Poisson's ratio in the plane strain constitutive matrix.*
To consider the opposite process, it is needed to consider the next expressions:

$$E^* = \frac{E(1+2\nu)}{(1+\nu)^2} \tag{0.9}$$

$$\nu^* = \frac{\nu}{1+\nu} \tag{0.10}$$

In the same way, it is needed to substitute the initial plane strain constitutive matrix :

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E^*}{(1+\nu^*)(1-2\nu^*)} \begin{bmatrix} 1-\nu^* & \nu^* & 0 \\ \nu^* & 1-\nu^* & 0 \\ 0 & 0 & \frac{1-2\nu^*}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix}$$

$$\frac{E(1+2\nu)}{(1+\nu)^2\left(1+\frac{\nu}{1+\nu}\right)\left(1-2\frac{\nu}{1+\nu}\right)} \begin{bmatrix} 1-\frac{\nu}{1+\nu} & \frac{\nu}{1+\nu} & 0 \\ \frac{\nu}{1+\nu} & 1-\frac{\nu}{1+\nu} & 0 \\ 0 & 0 & \frac{1}{2}\left(1-2\frac{\nu}{1+\nu}\right) \end{bmatrix}$$

$$\frac{E(1+2\nu)(1+\nu)(1+\nu)}{(1+\nu)^2(1+2\nu)(1-\nu)} \begin{bmatrix} \frac{1}{1+\nu} & \frac{\nu}{1+\nu} & 0 \\ \frac{\nu}{1+\nu} & \frac{1}{1+\nu} & 0 \\ 0 & 0 & \frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right) \end{bmatrix}$$

Finally the plane stress constitutive matrix is achieved:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix}$$

ASSIGNMENT 3.2

In the finite element formulation of near incompressible isotropic materials (as well as plasticity and viscoelasticity) it is convenient to use the so-called Lamé constants λ and μ instead of E and ν in the constitutive equations. Both λ and μ have the physical dimension of stress and are related to E and ν by:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{0.11}$$

$$\mu = G = \frac{E}{2(1+\nu)} \tag{0.12}$$

a) *Find the inverse relations for E, ν in terms of λ, ν.*
Solution a):

Now, to obtain a relationship of the mechanical properties in terms of the Lamé parameters, it is convenient to solve a system of equations by substituting some of the above equations, in that sense first the quotient of μ and λ is performed as:

$$\frac{\mu}{\lambda} = \frac{E/2(1+\nu)}{E\nu/(1+\nu)(1-2\nu)}$$

$$\frac{\mu}{\lambda} = \frac{1-2\nu}{2\nu}$$
$$2\nu\mu + 2\nu\lambda = \lambda$$
$$\nu = \frac{1}{2}\frac{\lambda}{\lambda+\mu}$$

Substituting ν in the equation related to G, we obtain:

$$\mu = G = \frac{E}{2(1+\nu)}$$
$$\mu = \frac{E}{2\left(1+\frac{1}{2}\frac{\lambda}{\lambda+\mu}\right)}$$
$$\mu = \frac{E}{2\left[(\lambda+\mu)+\lambda\right]/\lambda+\mu}$$
$$\mu = \frac{E(\lambda+\mu)}{3\lambda+2\mu}$$
$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$

Finally the mechanical properties in terms of the Lamé parameters are:

$$\nu = \frac{1}{2}\frac{\lambda}{\lambda+\mu} \qquad\qquad E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$

b) *Express the elastic matrix for plane stress and plane strain cases in terms of* λ,μ.

Solution b):
By using the relationship of mechanical properties and Lamé parameters, it is easy to obtain the constitutive plane stress matrix as:

$$\mathbf{E} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$
$$= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\frac{1}{1-\left(\frac{1}{2}\frac{\lambda}{\lambda+\mu}\right)^2}\begin{bmatrix} 1 & \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\frac{1}{2}\frac{\lambda}{\lambda+\mu}) \end{bmatrix}$$

$$= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \frac{1}{\left(1-\frac{1}{2}\frac{\lambda}{\lambda+\mu}\right)\left(1+\frac{1}{2}\frac{\lambda}{\lambda+\mu}\right)} \begin{bmatrix} 1 & \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 1 & 0 \\ 0 & 0 & \frac{\frac{2(\lambda+\mu)-\lambda}{2(\lambda+\mu)}}{2} \end{bmatrix}$$

$$= \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)} \begin{bmatrix} 1 & \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$= \frac{\mu}{\lambda+2\mu} \begin{bmatrix} 4(\lambda+\mu) & 2\lambda & 0 \\ 2\lambda & 4(\lambda+\mu) & 0 \\ 0 & 0 & \lambda+\mu \end{bmatrix}$$

Now, substituting in the plane strain constitutive matrix:

$$\mathbf{E} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$= \frac{\mu(3\lambda+2\mu)/(\lambda+\mu)}{(1+\frac{1}{2}\frac{\lambda}{\lambda+\mu})(1-2\frac{1}{2}\frac{\lambda}{\lambda+\mu})} \begin{bmatrix} 1-\frac{1}{2}\frac{\lambda}{\lambda+\mu} & \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 1-\frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ 0 & 0 & \frac{1-2\frac{1}{2}\frac{\lambda}{\lambda+\mu}}{2} \end{bmatrix}$$

$$= \frac{\mu(3\lambda+2\mu)/(\lambda+\mu)}{\frac{2(\lambda+\mu)+\lambda}{2(\lambda+\mu)}\frac{\lambda+\mu-\lambda}{\lambda+\mu}} \begin{bmatrix} \frac{2(\lambda+\mu)-\lambda}{2(\lambda+\mu)} & \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ \frac{1}{2}\frac{\lambda}{\lambda+\mu} & \frac{2(\lambda+\mu)-\lambda}{2(\lambda+\mu)} & 0 \\ 0 & 0 & \frac{\frac{(\lambda+\mu)-\lambda}{\lambda+\mu}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

c) *Split the stress-strain matrix E of plane strain as:*

$$\mathbf{E} = \mathbf{E}_\mu + \mathbf{E}_\lambda$$

in which E_μ and E_λ contain only μ and λ, respectively. This is the Lamé λ, μ splitting of the plane strain constitutive equations, which leads to the so-called B-bar formulation of near-incompressible finite elements.

Solution c):

Considering the plane-strain constitutive matrix obtained above:

$$\begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

$$= \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} + \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

d) *Express E_μ and E_λ also in terms of E and ν.*

Solution d):

By using the previous result, it is easy to substitute the values of ν and λ in terms of E and ν as:

$$\mathbf{E}_\mu = \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{E}_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

ASSIGNMENT 3.3

Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Let's consider for simplicity a = h = 1. The material parameters are E, ν. Initially ν is set to zero. Two structural models are considered for this problem as depicted in the figure:

- A plane linear Turner triangle with the same dimensions.
- A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.

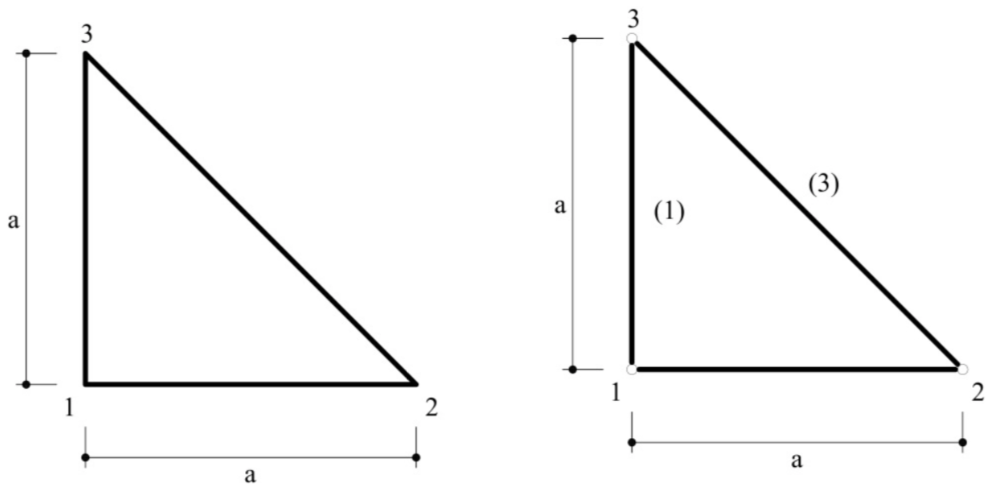

Figure 0.1: Problem Comparison: a) Turner triangle (left) - b) Three bars discretization (right)

***a)** Calculate the stiffness matrix K^e for both models.*
Solution a):

- Three Bars:

  The elemental global matrix for truss elements is:

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix} \tag{0.13}$$

  Now, particularizing the elemental global matrix for each bar of the problem:

  - Bar 1

* Nodes: 1 - 3
* $\theta = 90$
* $\cos(\theta) = 0$
* $\sin(\theta) = 1$

* Area: A_1
* Young Modulus: E
* Length: $a = 1$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x3} \\ f_{y3} \end{bmatrix} = EA_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x3} \\ u_{y3} \end{bmatrix} \tag{0.14}$$

- Bar 2

  * Nodes: 1 - 2
  * $\theta = 0$
  * $\cos(\theta) = \cos(0) = 1$
  * $\sin(\theta) = \sin(0) = 0$

  * Area: A_2
  * Young Modulus: E
  * Length: a

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = EA_2 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix} \tag{0.15}$$

- Bar 3

  * Nodes: 2 - 3
  * $\theta = (135)$
  * $\cos(\theta) = \cos(135) = -\sqrt{2}/2$
  * $\sin(\theta) = \sin(135) = \sqrt{2}/2$

  * Area: A_3
  * Young Modulus: E
  * Length: $a/\cos(45) = 2a/\sqrt{2}$

$$\begin{bmatrix} f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \frac{EA_3\sqrt{2}}{4} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \tag{0.16}$$

Expanding the three matrices to the total of degrees of freedom and then adding them, we obtain the global stiffness matrix of the system, the result is:

$$\mathbf{K}^e = E \begin{bmatrix} A_2 & 0 & -A_2 & 0 & 0 & 0 \\ & A_1 & 0 & 0 & 0 & -A_1 \\ & & A_2 + \frac{A_3\sqrt{2}}{4} & -\frac{A_3\sqrt{2}}{4} & -\frac{A_3\sqrt{2}}{4} & \frac{A_3\sqrt{2}}{4} \\ & & & \frac{A_3\sqrt{2}}{4} & \frac{A_3\sqrt{2}}{4} & -\frac{A_3\sqrt{2}}{4} \\ & SYM & & & \frac{A_3\sqrt{2}}{4} & -\frac{A_3\sqrt{2}}{4} \\ & & & & & A_1 + \frac{A_3\sqrt{2}}{4} \end{bmatrix} \tag{0.17}$$

- Turner Triangle:

  The element stiffness matrix of the "Turner Triangle" is defined as:

$$\mathbf{K}^e = \int_{\Omega^e} h\mathbf{B}^T\mathbf{E}\mathbf{B}\,d\Omega \tag{0.18}$$

As $\mathbf{B}, \mathbf{E}$ and A are constants, and also $h = 1$ and $a = 1$, the expression is integrated as:

$$\mathbf{K}^e = \frac{1}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{0.19}$$

where $x_{jk} = x_j - x_k$ and $y_{jk} = y_j - y_k$.

Node	X	Y
1	0.0	0.0
2	a=1.0	0.0
3	0.0	a=1.0

Table 0.1: Nodal coordinates.

Substituting the coordinates into the above equation:

$$\mathbf{K}^e = \frac{1}{4A}\begin{bmatrix} -1.0 & 0.0 & -1.0 \\ 0.0 & -1.0 & -1.0 \\ 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & 0.0 \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}\begin{bmatrix} -1.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ -1.0 & -1.0 & 0.0 & 1.0 & 1.0 & 0.0 \end{bmatrix}$$

Performing the corresponding matrix multiplications:

$$\mathbf{K}^e = \frac{1}{4A}\begin{bmatrix} E_{11}+2E_{13}+E_{33} & E_{12}+E_{13}+E_{23}+E_{33} & -E_{11}-E_{13} & -E_{13}-E_{33} & -E_{13}-E_{33} & -E_{12}-E_{23} \\ & E_{22}+2E_{23}+E_{33} & -E_{12}-E_{13} & -E_{23}-E_{33} & -E_{23}-E_{33} & -E_{22}-E_{23} \\ & & E_{11} & E_{13} & E_{13} & E_{12} \\ & & & E_{33} & E_{33} & E_{23} \\ & SYM & & & E_{33} & E_{23} \\ & & & & & E_{22} \end{bmatrix}$$

Considering the plane stress constitutive matrix:

$$\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{0.20}$$

considering initially $\nu = 0$ and substituting the corresponding values:

$$\mathbf{K}^e = \frac{E}{8A}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ & 3 & 0 & -1 & -1 & -2 \\ & & 2 & 0 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & SYM & & & 1 & 0 \\ & & & & & 2 \end{bmatrix} \tag{0.21}$$

b) *Is there any set of values for cross sections $A_1 = A_2 = A$ and $A_3 = A^{'}$ to make both stiffness matrix equivalent: $\mathbf{K_{bar}} = \mathbf{K_{triangle}}$? If not, which are these values to make them as similar as possible?*

Solution b):
Taking into account the cross sections equal for bar 1 and bar 2, the stiffness matrix of the first system is modified as next:

$$\mathbf{K}^e = E\begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ & A & 0 & 0 & 0 & -A \\ & & A+\frac{A^{'}\sqrt{2}}{4} & -\frac{A^{'}\sqrt{2}}{4} & -\frac{A^{'}\sqrt{2}}{4} & \frac{A^{'}\sqrt{2}}{4} \\ & & & \frac{A^{'}\sqrt{2}}{4} & \frac{A^{'}\sqrt{2}}{4} & -\frac{A^{'}\sqrt{2}}{4} \\ & SYM & & & \frac{A^{'}\sqrt{2}}{4} & -\frac{A^{'}\sqrt{2}}{4} \\ & & & & & A+\frac{A^{'}\sqrt{2}}{4} \end{bmatrix} \tag{0.22}$$

In order to obtain a matrix formulation composed by trusses that attempts to simulate the behavior of the Turner's continuum triangle, the procedure will consist in find cross sections of the bars that compensates the stiffness of the triangular element. Comparing equations 0.23 and 0.22, it can be seen that both are very different, in that sense the equations proposed to obtain these area values consist in compare the terms of the matrix diagonal, because of its importance giving the main rigidity to the system. The equations reduced to three possibilities[1]:

Case 1:

$$K_{11} \to EA = \frac{3E}{8A^*} \to A = \frac{3}{4}$$

$$K_{33} \to E\left(A + \frac{A^{'}\sqrt{2}}{4}\right) = \frac{2E}{8A^*} \to A^{'} = -\frac{\sqrt{2}}{2}$$

Case 2:

$$K_{11} \to A = \frac{3}{4}$$

$$K_{44} \to E\left(\frac{A^{'}\sqrt{2}}{4}\right) = \frac{E}{8A^*} \to A^{'} = \frac{\sqrt{2}}{2}$$

Case 3:

$$K_{44} \to E\left(\frac{A^{'}\sqrt{2}}{4}\right) = \frac{E}{8A^*} \to A^{'} = \frac{\sqrt{2}}{2}$$

[1] Note that $A^* = 1/2$ because it corresponds to the triangle area.

$$K_{33} \rightarrow E\left(A + \frac{A^{'}\sqrt{2}}{4}\right) = \frac{2E}{8A^{*}} \rightarrow A = \frac{1}{4}$$

It is interesting to observe that the first case results in a negative area $A^{'}$, so is not physically possible. The cases two and three give different possibilities of similitude between the bar formulation and the continuum triangle, *both are not equal*, but the diagonal terms are almost the same:

Case 2:

$$\mathbf{K}_{bar2} = \frac{E}{4}\begin{bmatrix} 3 & 0 & -3 & 0 & 0 & 0 \\ & 3 & 0 & 0 & 0 & -3 \\ & & 4 & -1 & -1 & 1 \\ & & & 1 & 1 & -1 \\ & SYM & & & 1 & -1 \\ & & & & & 1 \end{bmatrix}$$

Case 3:

$$\mathbf{K}_{bar3} = \frac{E}{4}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & -1 \\ & & 2 & -1 & -1 & 1 \\ & & & 1 & 1 & -1 \\ & SYM & & & 1 & -1 \\ & & & & & 1 \end{bmatrix}$$

Compared to the Turner's triangle:

$$\mathbf{K}^{e}_{triangle} = \frac{E}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ & 3 & 0 & -1 & -1 & -2 \\ & & 2 & 0 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & SYM & & & 1 & 0 \\ & & & & & 2 \end{bmatrix}$$

c) *Why these two stiffness matrix are not equivalent? Find a physical explanation.*

Solution c):

Each of the formulations pretends model different mechanical behaviors. As the bar (or truss in this case) is obtained by considering a linear element from one node to another. Its formulation it is based on consider material properties, as the elastic modulus E, and geometric, as the area A, and it can be placed in a global system due to the rotation matrix implied. This is a good analysis when the model presents lots of bars, e.g. the trusses of ceilings in construction.

The Turner's triangle is a different approach of analysis, by consider instead of a linear element, but a continuous element with the shape of a triangle. This element can be represented as a plate because it has a thickness, in comparison with the three bars triangle modeled by truss elements, which it is empty in the interior. Even though both analysis consider linear numerical approximation, they can not be completely taken as equal because of the type of approach used in their formulation.

The most relevant physical explanation of the difference behavior is because both elements works modeling different mechanical systems, as trusses (roofs ceilings, steel bridges, electrical towers, etc.) using the bar matrix, or continuum plates (walls, continuum beams, aeronautical parts, etc.) using the Turner's triangle.

d) *Solve question a) considering* $\nu \neq 0$ *and extract some conclusions*

Solution d):

As can be seen in the assembled matrix of the three bars system, there is no Poisson effect included because of its original formulation that is presented in the beginning of this assignment. In that sense, the only modified matrix corresponds to the Turner's triangle, that is presented next:

$$\mathbf{K}^e = \frac{E}{8A(1-\nu^2)} \begin{bmatrix} 3-\nu & 1+\nu & -2 & \nu-1 & \nu-1 & -2\nu \\ & 3-\nu & -2\nu & \nu-1 & \nu-1 & -2 \\ & & 2 & 0 & 0 & 2\nu \\ & & & 1-\nu & 1-\nu & 0 \\ & SYM & & & 1-\nu & 0 \\ & & & & & 2 \end{bmatrix} \tag{0.23}$$

As the equation above shows, the Poisson coefficient affects the stiffness increasing most of the components of the matrix due the value $\frac{1}{1-\nu^2}$ outside the integral. And also, the terms that forms a product with ν inside the matrix, instead of being zero now presents a value that increases the rigidity of that component.