Master of Science in Computational Mechanics

Computational Structural Mechanics and Dynamics

Assignment 3: Plane Stress and Strain

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Assignment 3.1

Suppose that the structural material is isotropic, with elastic modulus E and Poisson's ratio ν. The in-plane stress-strain relations for plane stress and plane strain as given in any textbook on elasticity are:

$$\textit{plane stress:} \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix}$$

$$\textit{plane strain:} \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix}$$

a) **Show that the constitutive matrix of plane strain can be formally obtained by replacing E by a fictitious modulus E^* and ν by a fictitious Poisson's ratio ν^* in the plane stress constitutive matrix. Find the expression of E^* and ν^* in terms of E and ν.**

By replacing the variables as explained in the statement of the problem, the following expression must hold:

$$\frac{E^*}{1-\nu^{*2}} \begin{bmatrix} 1 & \nu^* & 0 \\ \nu^* & 1 & 0 \\ 0 & 0 & \frac{1-\nu^*}{2} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \tag{1}$$

To achieve an element-to-element equivalency, the following expressions must be satisfied:

$$\begin{cases} \frac{E^*}{1-\nu^{*2}} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \\ \frac{E^*\nu^*}{1-\nu^{*2}} = \frac{E\nu}{(1+\nu)(1-2\nu)} \end{cases} \tag{2}$$

Dividing the second equation by the first one, it yields:

$$\nu^* = \frac{\cancel{E}\nu}{\cancel{(1+\nu)(1-2\nu)}} / \frac{\cancel{E}(1-\nu)}{\cancel{(1+\nu)(1-2\nu)}} = \frac{\nu}{1-\nu}$$

Substituting into the first equation, E^* can be found:

$$\begin{aligned} E^* &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}(1-\nu^{*2}) \\ &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(1-\left(\frac{\nu}{1-\nu}\right)^2\right) \\ &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \frac{(1-\nu)^2-\nu^2}{(1-\nu)^2} \\ &= \frac{E\cancel{(1-\nu)}}{(1+\nu)\cancel{(1-2\nu)}} \frac{\cancel{1-2\nu}}{(1-\nu)^{\cancel{2}}} = \frac{E}{1-\nu^2} \end{aligned}$$

Thus, the constitutive matrix for the plane stress and plane strain can be defined as:

$$\textit{plane stress:} \qquad \mathbf{E} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\textit{plane strain:} \qquad \mathbf{E} = \frac{E^*}{1-\nu^{*2}}\begin{bmatrix} 1 & \nu^* & 0 \\ \nu^* & 1 & 0 \\ 0 & 0 & \frac{1-\nu^*}{2} \end{bmatrix} \text{with} \begin{array}{l} \nu^* = \frac{\nu}{1-\nu} \\ E^* = \frac{E}{1-\nu^2} \end{array}$$

b) **Do also the inverse process: go from plane strain to plain stress by replacing a fictitious modulus and Poisson's ratio in the plane strain constitutive matrix.**

This device permits "reusing" a plane stress FEM program to do plane strain, or vice-versa, as long as the material is isotropic.

Using an analogous expression to equation (2), but by substituting E, ν for $\bar{E}$,$\bar{\nu}$ the following system of equations must be satisfied:

$$\begin{cases} \frac{E}{1-\nu^2} = \frac{\bar{E}(1-\bar{\nu})}{(1+\bar{\nu})(1-2\bar{\nu})} \\ \frac{E\nu}{1-\nu^2} = \frac{\bar{E}\bar{\nu}}{(1+\bar{\nu})(1-2\bar{\nu})} \end{cases} \tag{3}$$

Dividing the second equation by the first one, it yields:

$$\nu = \frac{\cancel{\bar{E}}\bar{\nu}}{\cancel{(1+\bar{\nu})(1-2\bar{\nu})}} \Big/ \frac{\cancel{\bar{E}}(1-\bar{\nu})}{\cancel{(1+\bar{\nu})(1-2\bar{\nu})}} = \frac{\bar{\nu}}{1-\bar{\nu}} \Rightarrow \bar{\nu} = \frac{\nu}{1+\nu}$$

Substituting into the first equation, $\bar{E}$ takes the value:

$$\frac{E\nu}{1-\nu^2} = \frac{\bar{E}\left(\frac{\nu}{1+\nu}\right)}{\left(1+\frac{\nu}{1+\nu}\right)\left(1-2\frac{\nu}{1+\nu}\right)}$$

$$\frac{E\nu}{1-\nu^2} = \frac{\bar{E}\left(\frac{\nu}{1+\nu}\right)}{\left(\frac{1+\nu+\nu}{1+\nu}\right)\left(\frac{1+\nu-2\nu}{1+\nu}\right)}$$

$$\frac{E\nu}{1-\nu^2} = \frac{\bar{E}\left(\frac{\nu}{1+\nu}\right)}{\left(\frac{1+2\nu}{1+\nu}\right)\left(\frac{1-\nu}{1+\nu}\right)}$$

$$\frac{E\cancel{\nu}}{\cancel{1-\nu^2}} = \bar{E}\frac{\cancel{\nu}(1+\nu)^2}{\cancel{(1+\nu)}(1+2\nu)\cancel{(1-\nu)}}$$

$$\Rightarrow \bar{E} = E\frac{1+2\nu}{(1+\nu)^2}$$

Thus, the constitutive matrix for the plane stress and plane strain can be defined as:

$$\textit{plane stress:} \qquad \mathbf{E} = \frac{\bar{E}}{(1+\bar{\nu})(1-2\bar{\nu})} \begin{bmatrix} 1-\bar{\nu} & \bar{\nu} & 0 \\ \bar{\nu} & 1-\bar{\nu} & 0 \\ 0 & 0 & \frac{1-2\bar{\nu}}{2} \end{bmatrix} \text{with} \quad \begin{array}{r} \bar{\nu} = \frac{\nu}{1+\nu} \\ \bar{E} = E\frac{1+2\nu}{(1+\nu)^2} \end{array}$$

$$\textit{plane strain:} \qquad \mathbf{E} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Assignment 3.2

In the finite element formulation of near incompressible isotropic materials (as well as plasticity and viscoelasticity) it is convenient to use the so-called Lamé constants λ and μ instead of E and ν in the constitutive equations. Both λ and μ have the physical dimension of stress and are related to E and ν by:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = G = \frac{E}{2(1+\nu)}$$

a) **Find the inverse relations for E,ν in terms of λ, μ.**

Using the definitions given, the expression μ/λ simplifies to:

$$\begin{aligned}
\frac{\mu}{\lambda} &= \frac{\frac{\cancel{E}}{2\cancel{(1+\nu)}}}{\frac{\cancel{E}\nu}{\cancel{(1+\nu)}(1-2\nu)}} \\
\frac{\mu}{\lambda} &= \frac{1-2\nu}{2\nu} \\
2\nu\mu &= (1-2\nu)\lambda \\
2\nu\mu &= \lambda - 2\nu\lambda \\
2\nu\mu + 2\nu\lambda &= \lambda \\
\nu &= \frac{1}{2}\frac{\lambda}{\lambda+\mu}
\end{aligned}$$

Replacing in the corresponding equation, E can be expressed as:

$$\begin{aligned}
\mu &= \frac{E}{2\left(1+\frac{1}{2}\frac{\lambda}{\lambda+\mu}\right)} \\
\mu &= \frac{E}{\frac{2(\lambda+\mu)+\lambda}{\lambda+\mu}} \\
\mu &= \frac{E}{\frac{3\lambda+2\mu}{\lambda+\mu}} \\
E &= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}
\end{aligned}$$

b) **Express the elastic matrix for plane stress and plane strain cases in terms of λ,μ.**

Using the previously obtained final expressions for E and ν, the constitutive matrix for the plane stress case can be rewritten as follows:

$$
\begin{aligned}
\mathbf{E} &= \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \\
&= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\frac{1}{1-\left(\frac{1}{2}\frac{\lambda}{\lambda+\mu}\right)^2}\begin{bmatrix} 1 & \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 1 & 0 \\ 0 & 0 & \frac{1-\frac{1}{2}\frac{\lambda}{\lambda+\mu}}{2} \end{bmatrix} \\
&= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\frac{1}{\left(1-\frac{1}{2}\frac{\lambda}{\lambda+\mu}\right)\left(1+\frac{1}{2}\frac{\lambda}{\lambda+\mu}\right)}\begin{bmatrix} 1 & \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 1 & 0 \\ 0 & 0 & \frac{\frac{2(\lambda+\mu)-\lambda}{2(\lambda+\mu)}}{2} \end{bmatrix} \\
&= \frac{\mu\cancel{(3\lambda+2\mu)}}{\cancel{\lambda+\mu}}\frac{4(\lambda+\mu)^{\cancel{2}}}{(\lambda+2\mu)\cancel{(3\lambda+2\mu)}}\begin{bmatrix} 1 & \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 1 & 0 \\ 0 & 0 & \frac{\frac{\cancel{\lambda+\mu}}{2(\cancel{\lambda+\mu})}}{2} \end{bmatrix} \\
&= \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)}\begin{bmatrix} 1 & \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \\
&= \frac{\mu}{\lambda+2\mu}\begin{bmatrix} 4(\lambda+\mu) & 2\lambda & 0 \\ 2\lambda & 4(\lambda+\mu) & 0 \\ 0 & 0 & \lambda+\mu \end{bmatrix}
\end{aligned}
$$

Similarly, for the plane strain problem the constitutive can be expressed in terms of λ and μ as:

$$
\begin{aligned}
\mathbf{E} &= \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \\
&= \frac{\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}}{(1+\frac{1}{2}\frac{\lambda}{\lambda+\mu})(1-2\frac{1}{2}\frac{\lambda}{\lambda+\mu})}\begin{bmatrix} 1-\frac{1}{2}\frac{\lambda}{\lambda+\mu} & \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 1-\frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ 0 & 0 & \frac{1-2\frac{1}{2}\frac{\lambda}{\lambda+\mu}}{2} \end{bmatrix} \\
&= \frac{\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}}{\frac{2(\lambda+\mu)+\lambda}{2(\lambda+\mu)}\frac{\lambda+\mu-\lambda}{\lambda+\mu}}\begin{bmatrix} \frac{2(\lambda+\mu)-\lambda}{2(\lambda+\mu)} & \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ \frac{1}{2}\frac{\lambda}{\lambda+\mu} & \frac{2(\lambda+\mu)-\lambda}{2(\lambda+\mu)} & 0 \\ 0 & 0 & \frac{\frac{(\lambda+\mu)-\lambda}{\lambda+\mu}}{2} \end{bmatrix} \\
&= \frac{2\cancel{\mu}\cancel{(3\lambda+2\mu)}(\lambda+\mu)^{\cancel{2}}}{\cancel{(\lambda+\mu)}\cancel{(3\lambda+2\mu)}\cancel{\mu}}\begin{bmatrix} \frac{2\mu+\lambda}{2(\lambda+\mu)} & \frac{1}{2}\frac{\lambda}{\lambda+\mu} & 0 \\ \frac{1}{2}\frac{\lambda}{\lambda+\mu} & \frac{2\mu+\lambda}{2(\lambda+\mu)} & 0 \\ 0 & 0 & \frac{\mu}{2(\lambda+\mu)} \end{bmatrix}
\end{aligned}
$$

$$
= \cancel{2(\lambda+\mu)} \begin{bmatrix} \frac{2\mu+\lambda}{\cancel{2(\lambda+\mu)}} & \frac{1}{\cancel{2}}\frac{\lambda}{\cancel{\lambda+\mu}} & 0 \\ \frac{1}{\cancel{2}}\frac{\lambda}{\cancel{\lambda+\mu}} & \frac{2\mu+\lambda}{\cancel{2(\lambda+\mu)}} & 0 \\ 0 & 0 & \frac{\mu}{\cancel{2(\lambda+\mu)}} \end{bmatrix}
$$

$$
= \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}
$$

c) **Split the stress-strain matrix E of plane strain as:**

$$
\mathbf{E} = \mathbf{E}_\mu + \mathbf{E}_\lambda
$$

in which E_μ and E_λ contain only μ and λ, respectively. This is the Lamé λ, μ splitting of the plane strain constitutive equations, which leads to the so-called B-bar formulation of near-incompressible finite elements.

Using the expression obtained in the previous section:

$$
\mathbf{E} = \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}
$$

$$
= \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} + \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
= \underbrace{\mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}_\mu} + \underbrace{\lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{E}_\lambda}
$$

d) **Express E_μ and E_λ also in terms of E and ν.**

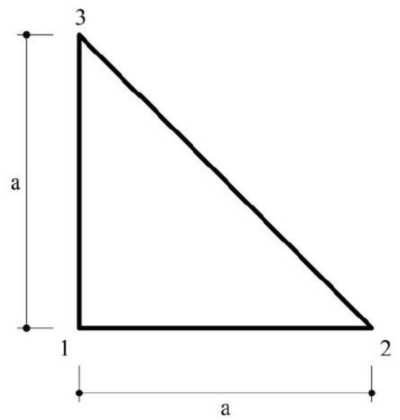
Giving the definition of μ and λ in terms of E and ν, E_μ and E_λ can be expressed as:

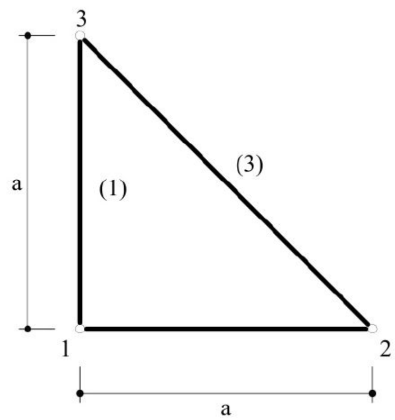
$$
\mathbf{E}_\mu = \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{E}_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

Assignment 3.3

Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Let's consider for simplicity $a = h = 1$. The material parameters are E, ν. Initially ν is set to zero. Two structural models are considered for this problem as depicted in the figure:

- **A plane linear Turner triangle with the same dimensions.**
- **A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.**





a) **Calculate the stiffness matrix $\mathbf{K^e}$ for both models.**

It will be considered that the global system of coordinates will be placed at node 1 for each one of the systems given.

Firstly, in the case of the Turner triangle, its element stiffness matrix is defined as:

$$\mathbf{K}^e = \int_{\Omega^e} h\mathbf{B}^T\mathbf{E}\mathbf{B}d\Omega$$

Since $\mathbf{B}, \mathbf{E}$ and A are constants, and taking into account that $h = a = 1$, the expression can be integrated directly and it becomes:

$$\mathbf{K}^e = \frac{1}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

where $x_{jk} = x_j - x_k$ and $y_{jk} = y_j - y_k$.

Considering the plane stress case with $\nu = 0$, the constitutive matrix reads:

$$\mathbf{E} = E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Since $A = 1/2$ and substituting the corresponding node coordinate values, the stiffness matrix simplifies to:

$$\mathbf{K}_{\mathbf{triangle}} = \frac{E}{4} \begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ & 3 & 0 & -1 & -1 & -2 \\ & & 2 & 0 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 0 \\ Symm. & & & & & 2 \end{bmatrix}$$

Secondly, for the set of three bar elements, the stiffness matrix of each bar element K^e can be written as:

$$K^e = \frac{EA^e}{L^{(e)}} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \tag{4}$$

where s, c are defined as the sine and cosine of the angle α between the local coordinate system of the bar with respect to the global coordinate system. For the given problem, α equals $\pi/2$, 0 and $-\pi/4$ for elements (1), (2) and (3), respectively. Thus,

$$K^{(1)} = \frac{EA_1}{a} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K^{(2)} = \frac{EA_2}{a} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^{(3)} = \frac{EA_3}{a\sqrt{2}} \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

Considering $A_1 = A_2 = A$, $A_3 = A'$, $a = 1$ and performing the corresponding assembly procedure, the global matrix becomes:

$$\mathbf{K_{bar}} = E \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ & A & 0 & 0 & 0 & -A \\ & & A + A'/2\sqrt{2} & -A'/2\sqrt{2} & -A'/2\sqrt{2} & A'/2\sqrt{2} \\ & & & A'/2\sqrt{2} & A'/2\sqrt{2} & -A'/2\sqrt{2} \\ & & & & A'/2\sqrt{2} & -A'/2\sqrt{2} \\ \textit{Symm.} & & & & & A + A'/2\sqrt{2} \end{bmatrix}$$

b) **Is there any set of values for cross sections $A_1 = A_2 = A$ and $A_3 = A'$ to make both stiffness matrix equivalent: $\mathbf{K}^{\mathbf{bar}} = \mathbf{K}^{\mathbf{triangle}}$? If not, which are these values to make them as similar as possible?**

At first sight, it becomes evident that it is impossible to make off-diagonal elements match. Nevertheless, it is possible to make the diagonal elements to coincide if the following conditions are imposed:

$$K_{11}^{bar} = K_{11}^{triangle} \qquad K_{33}^{bar} = K_{33}^{triangle} \qquad K_{44}^{bar} = K_{44}^{triangle}$$

It can be noticed that by imposing the previous conditions, the equivalence for elements K_{22}, K_{55} and K_{66} is automatically satisfied. Thus,

$$EA = E\tfrac{3}{4} \;\; \textit{(I)} \quad E(A + \tfrac{A'}{2\sqrt{2}}) = E\tfrac{2}{4} \;\; \textit{(II)} \quad E\tfrac{A'}{2\sqrt{2}} = \tfrac{E}{4} \;\; \textit{(III)}$$

Since the system is not solvable, the three following cases are mathematically possible:

Case 1	*Case 2*	*Case 3*
I and III	I and II	II and III
$A = 3/4$	$A = 3/4$	$A' = \sqrt{2}/2$
$A' = \sqrt{2}/2$	$A' = -\sqrt{2}/2$	$A = 1/4$

Neglecting case 2 for being physically impossible, two possibilities are considered: $A = 3/4$, $A' = \sqrt{2}/2$ and $A' = \sqrt{2}/2$, $A = 1/4$. It can be noticed that for both cases equivalence of the elements K_{45} will be satisfied. Hence, partial equivalence between the diagonal elements is achieved. For case 1:

$$\mathbf{K_{bar}} = E \begin{bmatrix} 3/4 & 0 & -3/4 & 0 & 0 & 0 \\ & 3/4 & 0 & 0 & 0 & -3/4 \\ & & 1 & -1/4 & -1/4 & 1/4 \\ & & & 1/4 & 1/4 & -1/4 \\ & & & & 1/4 & -1/4 \\ \textit{Symm.} & & & & & 1 \end{bmatrix}$$

$$\mathbf{K_{triangle}} = E \begin{bmatrix} 3/4 & 1/4 & -1/2 & -1/2 & -1/2 & 0 \\ & 3/4 & 0 & -1/4 & -1/4 & -1/2 \\ & & 1/2 & 0 & 0 & 0 \\ & & & 1/4 & 1/4 & 0 \\ & & & & 1/4 & 0 \\ \textit{Symm.} & & & & & 1/2 \end{bmatrix}$$

Furthermore, for case 3:

$$\mathbf{K_{bar}} = E \begin{bmatrix} 1/4 & 0 & -1/4 & 0 & 0 & 0 \\ & 1/4 & 0 & 0 & 0 & -1/4 \\ & & 1/2 & -1/4 & -1/4 & 1/4 \\ & & & 1/4 & 1/4 & -1/4 \\ & & & & 1/4 & -1/4 \\ \mathit{Symm.} & & & & & 1/2 \end{bmatrix}$$

$$\mathbf{K_{triangle}} = E \begin{bmatrix} 3/4 & 1/4 & -1/2 & -1/2 & -1/2 & 0 \\ & 3/4 & 0 & -1/4 & -1/4 & -1/2 \\ & & 1/2 & 0 & 0 & 0 \\ & & & 1/4 & 1/4 & 0 \\ & & & & 1/4 & 0 \\ \mathit{Symm.} & & & & & 1/2 \end{bmatrix}$$

c) **Why these two stiffness matrix are not equivalent? Find a physical explanation.**

In order to answer this question, one might consider the case 1 of the previous question (i.e. equivalence of elements K_{11} and K_{44}) and an applied horizontal force f at node 1. Then, the first equation of the systems will read:

$$\begin{aligned} \mathit{Bars:} \quad & \frac{3}{4}u_{1x} - \frac{3}{4}u_{2x} = f \\ \mathit{Triangle:} \quad & \frac{3}{4}u_{1x} + \frac{1}{4}u_{1y} - \frac{1}{2}u_{2x} - \frac{1}{2}u_{2y} - \frac{1}{2}u_{3x} = f \end{aligned}$$

Understanding the stiffness matrix as a measure of the resistance to deformation that a mechanical system possess, it can be seen that in the case of the bars, the force can only be 'shared' between node 1 and 2 since they are horizontally connected whereas node 3 cannot contribute because it is vertically connected to node 1. On the other hand and since the triangle element is by definition continuous, the force can be 'shared' by all nodes and, thus, the resistance comes from all the nodes altogether.

Additionally, it is relevant to mention that the nature of the elements also provide some insights to understand the non-equivalence of the stiffness matrices. Bar elements only transmit forces axially and the ability of the nodes to 'share' the external forces depends on the connection between them. Contrarily, the triangle elements as a continuum allows the nodes to resist external forces in a combined way.

d) **Solve question a) considering $\nu \neq 0$ and extract some conclusions.**

Considering $\nu \neq 0$, the stiffness matrix remains unchanged and, as a result, the stiffness matrix of triangle becomes:

$$\mathbf{K_{triangle}} = \frac{E}{4(1-\nu^2)} \begin{bmatrix} 3-\nu & 1+\nu & -2 & \nu-1 & \nu-1 & -2\nu \\ & 3-\nu & -2\nu & \nu-1 & \nu-1 & -2 \\ & & 2 & 0 & 0 & 2\nu \\ & & & 1-\nu & 1-\nu & 0 \\ & & & & 1-\nu & 0 \\ Sym. & & & & & 2 \end{bmatrix}$$

In this case, equivalence between the stiffness matrices would be even more difficult to achieve and also, because of the physical meaning the Poisson's ratio ν, a horizontal force could cause both horizontal and vertical displacements.