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Assignment 3.1

Given: the Relation of λ and μ to E and ν

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = G = \frac{E}{2(1+\nu)}$$

① $\Rightarrow$

we have

$$\mu = \frac{E}{2(1+\nu)}$$

$$E = 2\mu(1+\nu) \quad \ldots ①$$

we also have

$$\lambda = \frac{E\,\nu}{(1+\nu)\,(1-2\nu)}$$

substituting E from equation ①

$$\lambda = \frac{2\mu\cancel{(1+\nu)}\,\nu}{\cancel{(1+\nu)}\,(1-2\nu)}$$

$$\Rightarrow \lambda(1-2\nu) = 2\mu\nu$$

$$\Rightarrow \lambda - 2\lambda\nu = 2\mu\nu$$

$$\Rightarrow 2\mu\nu + 2\lambda\nu = \lambda$$

$$\Rightarrow \boxed{\nu = \frac{\lambda}{2(\mu+\lambda)}}$$

Substituting the value of ν in equation (1)

$$E = 2\mu\left(1 + \frac{\lambda}{2(\mu+\lambda)}\right)$$

$$= 2\mu \frac{(2\mu + 2\lambda + \lambda)}{2(\mu+\lambda)}$$

$$\boxed{E = \frac{\mu(2\mu + 3\lambda)}{(\mu+\lambda)}}$$

(2)

we Known, from plane stress

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

$$= \frac{E}{(1-\nu)(1+\nu)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

$$(1-\nu) = 1 - \frac{\lambda}{2(\mu+\lambda)} = \frac{2\mu + 2\lambda - \lambda}{2(\mu+\lambda)} = \frac{2\mu+\lambda}{2(\mu+\lambda)}$$

$$(1+\nu) = 1 + \frac{\lambda}{2(\mu+\lambda)} = \frac{2\mu + 2\lambda + \lambda}{2(\mu+\lambda)} = \frac{2\mu + 3\lambda}{2(\mu+\lambda)}$$

③ we have

The, Elastic matrix for plane stress

$$\frac{E}{(1-\nu)(1+\nu)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$= \frac{\mu\cancel{(2\mu+3\lambda)}}{\cancel{(\mu+\lambda)}} \times \frac{2\cancel{(\mu+\lambda)}}{(2\mu+\lambda)} \times \frac{2(\mu+\lambda)}{\cancel{(2\mu+3\lambda)}}\begin{bmatrix} 1 & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & 1 & 0 \\ 0 & 0 & \frac{2\mu+\lambda}{4(\mu+\lambda)} \end{bmatrix}$$

$$= \frac{4(\mu+\lambda)\mu}{(2\mu+\lambda)}\begin{bmatrix} 1 & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & 1 & 0 \\ 0 & 0 & \frac{2\mu+\lambda}{4(\mu+\lambda)} \end{bmatrix}$$

$$= \frac{\mu}{(2\mu+\lambda)}\begin{bmatrix} 4(\mu+\lambda) & 2\lambda & 0 \\ 2\lambda & 4(\mu+\lambda) & 0 \\ 0 & 0 & 2\mu+\lambda \end{bmatrix}$$

We know, from plane strain

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

The elastic matrix for plane strain

$$\Rightarrow \quad \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

$$(1-2\nu) = 1 - 2\frac{\lambda}{2(\mu+\lambda)} = \frac{\mu + \cancel{\lambda} - \cancel{\lambda}}{(\mu+\lambda)} = \frac{\mu}{(\mu+\lambda)}$$

$$\Rightarrow \frac{\mu\cancel{(2\mu+3\lambda)}}{\cancel{(\mu+\lambda)}} \times \frac{2\cancel{(\mu+\lambda)}}{\cancel{(2\mu+3\lambda)}} \times \frac{(\mu+\lambda)}{\cancel{\mu}} \begin{bmatrix} \frac{2\mu+\lambda}{2(\mu+\lambda)} & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & \frac{2\mu+\lambda}{2(\mu+\lambda)} & 0 \\ 0 & 0 & \frac{\mu}{2(\mu+\lambda)} \end{bmatrix}$$

$$\Rightarrow 2\cancel{(\mu+\lambda)} \frac{1}{2\cancel{(\mu+\lambda)}} \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

$$= \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

(3) => Stress-strain matrix $\underline{E}$ in terms of $\underline{E_\lambda}$ & $\underline{E_\mu}$

$$\underline{E} = \underline{E_\mu} + \underline{E_\lambda}$$

$$\begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} + \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{E_\lambda} = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{E_\mu} = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

(6)

(4) $\Rightarrow$

$$\underline{E_\lambda} = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{E_\mu} = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

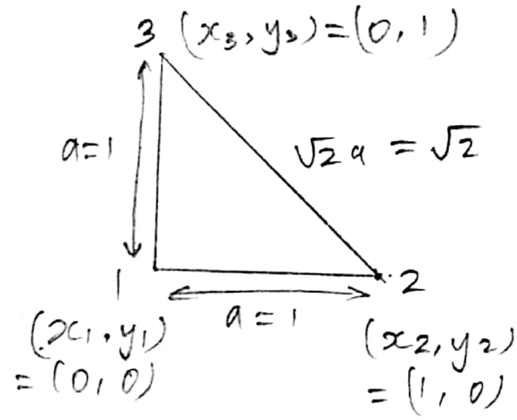
$$= \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Assignment 3.2

① ⇒ need to calculate the stiffness matrices K_{tri} and K_{bar} for both discrete models.

⇒ the plane linear Turner Triangle



ⓐ Element Stiffness Matrix

$$K^e = \int_{\Omega^e} h \underline{B}^T \underline{E} \underline{B} \, d\Omega \quad (\underline{B} \text{ and } \underline{E} \text{ are constant over } \Omega^e)$$

$$= \underline{B}^T \underline{E} \underline{B} \int_{\Omega^e} h \, d\Omega$$

$$K^e = \frac{h}{4A} \underline{B}^T \underline{E} \underline{B} \qquad \left[\because \int_{\Omega^e} h \, d\Omega = hA \text{ since } h \text{ is uniform} \right]$$

ⓑ Area (A):

$$2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$2A = 1$$

$$A = \frac{1}{2}$$

(2)

(i) strain displacement (B)

we know kinematic equations

$$e = DNu^e = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

$$= \underline{\underline{B}}\, u^e$$

where, $y_{jk} = y_j - y_k$ & $x_{jk} = x_j - x_k$

$y_{23} = -1$; $x_{32} = -1$;
$y_{31} = 1$; $x_{13} = 0$
$y_{12} = 0$; $x_{21} = 1$

$$\underline{\underline{B}} = \frac{1}{2\times\frac{1}{2}}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(ii) Constitutive matrix ($\underline{\underline{E}}$)

from Plane stress:

$$\underline{\sigma} = \underline{\underline{E}}\,\underline{e} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

we have $\nu = 0$:

$$\underline{\underline{E}} = E\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

③

Now,

Solving for stiffness matrix

$$K^e_{tri} = \frac{h}{4A}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \times E \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

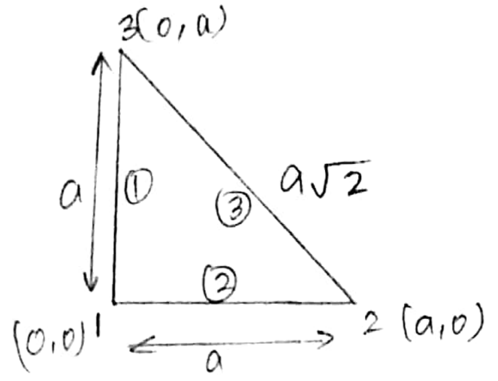
we have, $h = 1$; $A = 1/2$

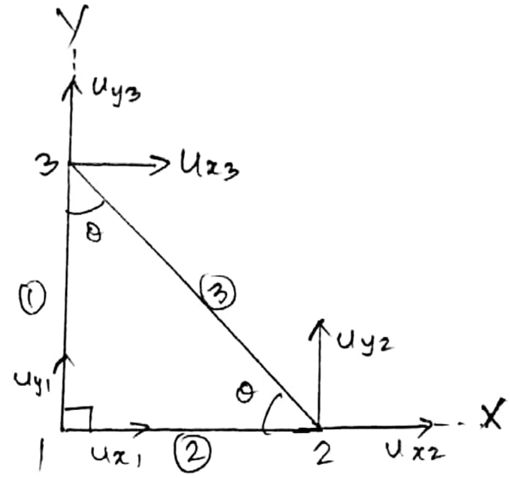
$$K^e_{tri} = \frac{1}{2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \times E \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$K^e_{tri} = E\begin{bmatrix} 0.75 & 0.25 & -0.5 & -0.25 & -0.25 & 0 \\ 0.25 & 0.75 & 0 & -0.25 & -0.25 & -0.5 \\ -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ -0.25 & -0.25 & 0 & 0.25 & 0.25 & 0 \\ -0.25 & -0.25 & 0 & 0.25 & 0.25 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

(1)

⇒ Three bar element





$\theta = 45$

ⓐ Element Stiffness Matrix

$$K^e = (T^e)^T K^e (T^e)$$

$$K^e = \frac{E^e A^e}{l^e} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \quad \cdots (1)$$

$c \Rightarrow \cos\theta \quad ; \quad s \Rightarrow \sin\theta$

ⓑ Element wise stiffness matrix

for **Element 1** $\quad \theta = 90°$, $A^e = A_1$ & $l_1 = a$

$$C_1 = \cos\theta = \cos 90 = 0$$

$$S_1 = \sin\theta = \sin 90 = 1$$

$$K_1 = \frac{E A_1}{l_1} \begin{bmatrix} c_1^2 & c_1 s_1 & -c_1^2 & -c_1 s_1 \\ c_1 s_1 & s_1^2 & -c_1 s_1 & -s_1^2 \\ -c_1^2 & -c_1 s_1 & c_1^2 & c_1 s_1 \\ -c_1 s_1 & -s_1^2 & c_1 s_1 & s_1^2 \end{bmatrix}$$

Substituting the values.

$$K_1 = \frac{EA_1}{a}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad \ldots (2)$$

For Element 2

$\theta = 0°$, $A^e = A_2$, $l_2 = a$

$C_2 = \cos\theta = \cos 0 = 1$

$S_2 = \sin\theta = \sin 0 = 0$

$$K_2 = \frac{EA_2}{l_2}\begin{bmatrix} C_2^2 & C_2S_2 & -C_2^2 & -C_2S_2 \\ C_2S_2 & S_2^2 & -C_2S_2 & -S_2^2 \\ -C_2^2 & -C_2S_2 & C_2^2 & C_2S_2 \\ -C_2S_2 & -S_2^2 & C_2S_2 & S_2^2 \end{bmatrix}$$

Substituting the values

$$K_2 = \frac{EA_2}{a}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \ldots (3)$$

for element 3

$\theta = 135^\circ \quad , A' = A_3 \quad l_3 = a\sqrt{2}$

$C_3 = \cos 135 = -1/\sqrt{2}$

$S_3 = \sin 135 = 1/\sqrt{2}$

$$K_3 = \frac{EA_3}{l_3}\begin{bmatrix} C_3^2 & C_3S_3 & -C_3^2 & -C_3S_3 \\ C_3S_3 & S_3^2 & -C_3S_3 & -S_3^2 \\ -C_3^2 & -C_3S_3 & C_3^2 & C_3S_3 \\ -C_3S_3 & -S_3^2 & C_3S_3 & S_3^2 \end{bmatrix}$$

$$K_3 = \frac{EA_3}{a\sqrt{2}}\begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

$$= \frac{EA_3}{a\,2\sqrt{2}}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

(7)

The globalized stiffness matrix

$$K_{bar} = K_1 + K_2 + K_3$$

$$K_{bar} = \begin{bmatrix} \frac{EA_2}{a} & 0 & -\frac{EA_2}{a} & 0 & 0 & 0 \\ 0 & \frac{EA_1}{a} & 0 & 0 & 0 & -\frac{EA_1}{a} \\ -\frac{EA_2}{a} & 0 & \frac{EA_2}{a}+\frac{EA_3}{2\sqrt{2}a} & -\frac{EA_3}{2\sqrt{2}a} & -\frac{EA_3}{2\sqrt{2}a} & \frac{EA_3}{2\sqrt{2}a} \\ 0 & 0 & -\frac{EA_3}{2\sqrt{2}a} & \frac{EA_3}{2\sqrt{2}a} & \frac{EA_3}{2\sqrt{2}a} & -\frac{EA_3}{2\sqrt{2}a} \\ 0 & 0 & -\frac{EA_3}{2\sqrt{2}a} & \frac{EA_3}{2\sqrt{2}a} & \frac{EA_3}{2\sqrt{2}a} & -\frac{EA_3}{2\sqrt{2}a} \\ 0 & -\frac{EA_1}{a} & \frac{EA_3}{2\sqrt{2}a} & -\frac{EA_3}{2\sqrt{2}a} & -\frac{EA_3}{2\sqrt{2}a} & \frac{EA_1}{a}+\frac{EA_3}{2\sqrt{2}a} \end{bmatrix}$$

Substituting a as 1

$$K_{bar} = \begin{bmatrix} EA_2 & 0 & -EA_2 & 0 & 0 & 0 \\ 0 & EA_1 & 0 & 0 & 0 & -EA_1 \\ -EA_2 & 0 & EA_2+\frac{EA_3}{2\sqrt{2}} & -\frac{EA_3}{2\sqrt{2}} & -\frac{EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} & -\frac{EA_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} & -\frac{EA_3}{2\sqrt{2}} \\ 0 & -EA_1 & \frac{EA_3}{2\sqrt{2}} & -\frac{EA_3}{2\sqrt{2}} & -\frac{EA_3}{2\sqrt{2}} & EA_1+\frac{EA_3}{2\sqrt{2}} \end{bmatrix}$$

while

$$K_{tri} = \begin{bmatrix} 0.75E & 0.25E & -0.5E & -0.25E & -0.25E & 0 \\ 0.25E & 0.75E & 0 & -0.25E & -0.25E & -0.5E \\ -0.5E & 0 & 0.5E & 0 & 0 & 0 \\ -0.25E & -0.25E & 0 & 0.25E & 0.25E & 0 \\ -0.25E & -0.25E & 0 & 0.25E & 0.25E & 0 \\ 0 & -0.5E & 0 & 0 & 0 & 0.5E \end{bmatrix}$$

(2) ⇒ No, there are no any set of values for the Cross-sections $A_1 = A_2$ and A_3 to make both stiffness matrix equivalent.

To make them more similar we need to consider $A_1 = A_2 = 0.5$ and $A_3 = 0$.

(3) ⇒ The two stiffness matrices are not equal because the displacement at any interior point while using triangular element is a linear combination of Shape functions. And we consider it to form the stiffness matrix using the constitutive relationship.

But for bar elements, it does not provide information on displacements of interior points. We can only recover axial stresses along bar axis.

(2) $\Rightarrow$ For the plane stress problems, when $\nu \neq 0$

$$\epsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \neq 0$$ (For isotropic materials)

Here, Due to the poisson's ratio effect, the transverse strain will be non-zero.

For the plane strain problems, when $\nu \neq 0$

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \neq 0$$ (for isotropic materials)

Because of the poisson's ratio effect, the transverse stress will become non-zero