Master Of Science in Computational Mechanics

Computational Structural Mechanics and Dynamics

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Assignment 3

Assignment 3.1

Suppose that the structural material is isotropic, with elastic modulus E and Poissons ratio . The in-plane stress-strain relations for plane stress and plane strain as given in any textbook on elasticity are

$$planestress: \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix} \tag{1}$$

$$planestrain: \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix} \tag{2}$$

a) Show that the constitutive matrix of plane strain can be formally obtained by replacing E by fictitious modulus E^* and ν by a fictitious Poissons ratio ν^* in the plane stress constitutive matrix. Find the expression of E^* and ν^* in terms of E and ν.

Replacing $E \rightarrow E^*$ and $\nu \rightarrow \nu^*$ in plane stress constitutive matrix we get following,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E^*}{1-\nu^{*2}} & \frac{\nu^* E^*}{1-\nu^{*2}} & 0 \\ \frac{\nu^* E^*}{1-\nu^{*2}} & \frac{E^*}{1-\nu^{*2}} & 0 \\ 0 & 0 & \frac{E^*}{2(1+\nu^*)} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix} \tag{3}$$

for getting plane strain matrix, every term of the obtained matrix (3) must be equal to corresponding term of equation (2), therefore equating individual term (1,1) of each matrix,

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = \frac{E^*}{1-\nu^{*2}}$$

$$E^* = \frac{E(1-\nu)(1-\nu^*)(1+\nu^*)}{(1+\nu)(1-2\nu)}$$

when we equate (3,3) term of matrices of equation (2) and (3) we get,

$$\frac{E}{2(1+\nu)} = \frac{E^*}{2(1+\nu^*}$$

$$E^* = \frac{E(1+\nu}{(1+\nu)}$$

therefore,

$$\frac{E(1+\nu}{(1+\nu)} = \frac{E(1-\nu)(1-\nu^*)(1+\nu^*)}{(1+\nu)(1-2\nu)}$$

$$\boxed{\nu^* = \frac{\nu}{(1-\nu}}$$

$$\boxed{E^* = \frac{E}{1-\nu^2}}$$

b) Do also the inverse process: go from plane strain to plain stress by replacing a fictitious modulus and Poissons ratio in the plane strain constitutive matrix. This device permits reusing a plane stress FEM program to do plane strain, or vice-versa, as long as the material is isotropic.

Replacing $E \rightarrow E^*$ and $\nu \rightarrow \nu^*$ in plane strain constitutive matrix we get following,

$$planestrain: \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E^*(1-\nu)}{(1+\nu^*)(1-2\nu^*)} & \frac{E^*\nu^*)}{(1+\nu^*)(1-2\nu^*)} & 0 \\ \frac{E^*\nu^*)}{(1+\nu^*)(1-2\nu^*)} & \frac{E^*(1-\nu)}{(1+\nu^*)(1-2\nu^*)} & 0 \\ 0 & 0 & \frac{E^*}{2(1+\nu^*)} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix} \tag{4}$$

following similar steps we get for term(1,1)

$$\frac{E^*(1-\nu)}{(1+\nu^*)(1-2\nu^*)} = \frac{E}{1-\nu^2}$$

and for term (3,3)

$$\nu^* = \frac{E^*(1-\nu)-E}{E}$$

therefore,

$$\frac{E^*(\frac{E-E^*(1-\nu)-E}{E})}{(\frac{E+E^*(1-\nu)-E}{E})(\frac{E-2E^*(1-\nu)-2E}{E})} = \frac{E}{1-\nu^2}$$

$$\boxed{E^* = \frac{E(1-2\nu)}{(1+\nu)^2}}$$

so we get,

$$\boxed{\nu^* = \frac{-3\nu}{1+\nu}}$$

Assignment 3.2

In the finite element formulation of near incompressible isotropic materials (as well as plasticity and viscoelasticity) it is convenient to use the so-called Lam constants and instead of E and in the constitutive equations. Both and have the physical dimension of stress and are related to E and by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{5}$$

$$\mu = G = \frac{E}{2(1+\nu)} \tag{6}$$

a) Find the inverse relations for E, in terms of λ, μ
from above equations (5) and(6) we can infer,

$$E = \frac{\lambda(1+\nu)(1-2\nu)}{\nu}$$

and

$$E = 2\mu(1+\nu)$$

equating above equations we get,

$$\frac{\lambda(1+\nu)(1-2\nu)}{\nu} = 2\mu(1+\nu)$$

$$\boxed{\nu = \frac{\lambda}{2(\mu+\lambda)}}$$

$$\boxed{E = \frac{\mu(2\mu+3\lambda)}{(\mu+\lambda)}}$$

b) Express the elastic matrix for plane stress and plane strain cases in terms of λ, μ
For Plane Stress elastic matrix will be,

$$\frac{\frac{\mu(2\mu+3\lambda)}{(\mu+\lambda)}}{1-(\frac{\lambda}{2(\mu+\lambda)})^2}\begin{bmatrix} 1 & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & 1 & 0 \\ 0 & 0 & \frac{1-(\frac{\lambda}{2(\mu+\lambda)})}{2} \end{bmatrix}$$

$$\boxed{\frac{4\mu(\mu+\lambda)}{2\mu+\lambda}\begin{bmatrix} 1 & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & 1 & 0 \\ 0 & 0 & \frac{\lambda}{2(\mu+\lambda)} \end{bmatrix}}$$

And for Plane Strain elastic matrix will be,

$$\frac{\frac{\mu(2\mu+3\lambda)}{(\mu+\lambda)}}{(1+\frac{\lambda}{2(\mu+\lambda)})(1-2(\frac{\lambda}{2(\mu+\lambda)})}\begin{bmatrix} 1-(\frac{\lambda}{2(\mu+\lambda)}) & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & 1-(\frac{\lambda}{2(\mu+\lambda)}) & 0 \\ 0 & 0 & \frac{1-2(\frac{\lambda}{2(\mu+\lambda)})}{2} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}}$$

c) Split the stress-strain matrix E of plane strain as $E = E_\lambda + E_\mu$

Stress- Strain matrix E of Plane Strain is,

$$\begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

therefore,

$$E_\lambda + E_\mu = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} + \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

d) Express E_λ and E_μ also in terms of E and ν

$$\boxed{E_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$\boxed{E_\mu = \frac{E}{2(1+\nu)}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

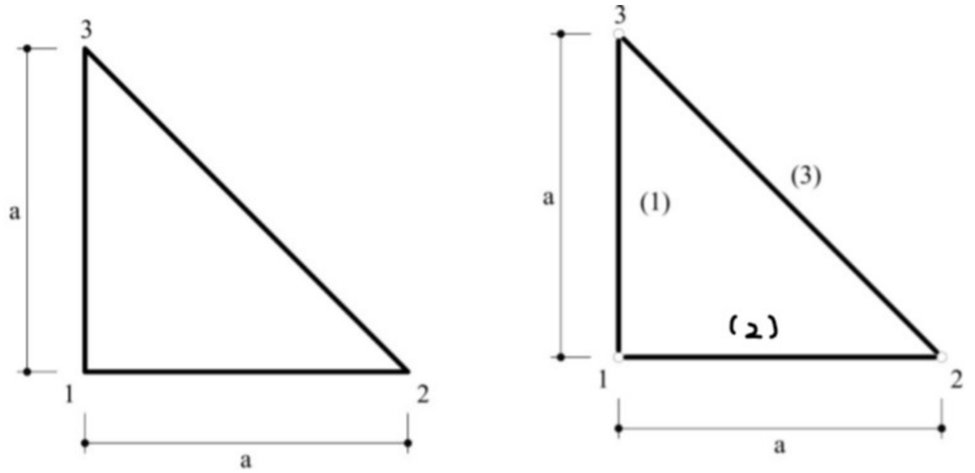

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Figure 1

Assignment 3.3

Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Lets consider for simplicity $a = h = 1$. The material parameters are E, . Initially is set to zero. Two structural models are considered for this problem as depicted in the figure 1:

- A plane linear Turner triangle with the same dimensions.
- A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.

a) Calculate the stiffness matrix K^e for both models.

- For Turner triangle:

Node	X	Y
1	0	0
2	1	0
3	0	1

Nodal co-ordinates

Area of triangle,

$$2A = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$2A = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \rightarrow A = \frac{1}{2}$$

Stiffness matrix for Turner Triangle,

$$K^e = \int_{\Omega}^{*} hB^T EBd\Omega$$

$$\mathbf{K}^e = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

as, $b_i = y_j - y_k$ and $c_i = x_k - x_j$, where i, j, k $\leftrightarrow 1, 2, 3$.
$b_1 = -1, b_2 = 1, b_3 = 0$
$c_1 = -1, c_2 = 0, c_3 = 1$
Therefore Stiffnessmatrix becomes,

$$K^e = \frac{1}{u\frac{1}{2}}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & \frac{E}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\boxed{k_{triangle} = \frac{1}{2}\begin{bmatrix} \frac{3E}{2} & \frac{E}{2} & -E & \frac{-E}{2} & \frac{-E}{2} & 0 \\ \frac{E}{2} & \frac{3E}{2} & 0 & \frac{-E}{2} & \frac{-E}{2} & -E \\ -E & 0 & E & 0 & 0 & 0 \\ \frac{-E}{2} & \frac{-E}{2} & 0 & \frac{E}{2} & \frac{E}{2} & 0 \\ \frac{-E}{2} & \frac{-E}{2} & 0 & \frac{E}{2} & \frac{E}{2} & 0 \\ 0 & -E & 0 & 0 & 0 & E \end{bmatrix}}$$

- For Bar Element,

Element stiffness matrix,

$$k^1 = \frac{E^1 A^1}{L^1}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & sc & c^2 & sc \\ -sc & s^2 & sc & s^2 \end{bmatrix}$$

$c = cos(\theta), s = sin(\theta)$

Element 1:=$\theta = 90$ and $L = 1$,

$$K^1 = EA_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & EA_1 & 0 & -EA_1 \\ 0 & 0 & 0 & 0 \\ 0 & -EA_1 & 0 & EA_1 \end{bmatrix}$$

Element 2:=$\theta = 0$ and $L = 1$,

$$K^2 = EA_2 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} EA_2 & 0 & -EA_2 & 0 \\ 0 & 0 & 0 & 0 \\ -EA_2 & 0 & EA_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 3:=$\theta = 45$ and $L = \sqrt{2}$,

$$K^3 = EA_3 \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} \frac{0.5EA_3}{\sqrt{2}} & \frac{-0.5EA_3}{\sqrt{2}} & \frac{-0.5EA}{\sqrt{2}} & \frac{0.5EA_3}{\sqrt{2}} \\ \frac{-0.5EA_3}{\sqrt{2}} & \frac{0.5EA_3}{\sqrt{2}} & \frac{0.5EA_3}{\sqrt{2}} & \frac{-0.5EA_3}{\sqrt{2}} \\ \frac{-0.5EA_3}{\sqrt{2}} & \frac{0.5EA_3}{\sqrt{2}} & \frac{0.5EA_3}{\sqrt{2}} & \frac{-0.5EA_3}{\sqrt{2}} \\ \frac{0.5EA_3}{\sqrt{2}} & \frac{-0.5EA_3}{\sqrt{2}} & \frac{-0.5EA_3}{\sqrt{2}} & \frac{0.5EA_3}{\sqrt{2}} \end{bmatrix}$$

Global stiffness matrix,

$$\boxed{K_{bar} = \begin{bmatrix} EA_2 & 0 & -EA_2 & 0 & 0 & 0 \\ 0 & EA_1 & 0 & 0 & 0 & EA_1 \\ -EA_2 & 0 & EA_2 + \frac{0.5EA_3}{\sqrt{2}} & -\frac{-0.5EA_3}{\sqrt{2}} & -\frac{-0.5EA}{\sqrt{2}} & \frac{0.5EA_3}{\sqrt{2}} \\ 0 & 0 & -\frac{-0.5EA_3}{\sqrt{2}} & \frac{0.5EA_3}{\sqrt{2}} & \frac{0.5EA_3}{\sqrt{2}} & -\frac{-0.5EA_3}{\sqrt{2}} \\ 0 & 0 & -\frac{-0.5EA_3}{\sqrt{2}} & \frac{0.5EA_3}{\sqrt{2}} & \frac{0.5EA_3}{\sqrt{2}} & -\frac{-0.5EA_3}{\sqrt{2}} \\ 0 & -EA_1 & \frac{0.5EA_3}{\sqrt{2}} & -\frac{-0.5EA_3}{\sqrt{2}} & -\frac{-0.5EA_3}{\sqrt{2}} & EA_1 + \frac{0.5EA_3}{\sqrt{2}} \end{bmatrix}}$$

b) Is there any set of values for cross sections $A_1 = A_2 = A$ and $A_3 = A'$ to make both stiffness matrix equivalent: $K_b ar = K_{triangle}$? If not, which are these values to make them as similar as possible?

→ Specifically all the three areas are arbitrary so if at all we have to put specific values the most can be done is make diagonal element of the stiffness matrices equal. Which roughly calculates to $A_1 = A_2 = 73/4$ and $A_3 = 1/\sqrt{2}$.

c) Why these two stiffness matrix are not equivalent? Find a physical explanation.

→ The basic difference between bar and tr angular element is the structure, bar element represents only boundary of the triangle while triangular element represents whole 2D surface of triangle. Which makes bar element less stiff then the triangular element. It can be also seen from the stiffness matrices as bar the one with bar element has more zeros then triangular.

d) Solve question a) considering $\nu \neq 0$ and extract some conclusions.

Thus stiffness matrix will be,

$$\frac{E}{1-\nu}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$K_{triangle} = \begin{bmatrix} \frac{3-\nu}{2} & \frac{\nu+1}{2} & -1 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -\nu \\ \frac{\nu+1}{2} & \frac{3-\nu}{2} & -\nu & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

as Poisson's ratio is introduced inn the system the system becomes more stiff as the number of zero elements tn stiffness matrix are reduced.