Computational Structural Mechanics and Dynamics

Assignment 3

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Assignment 3.1

Suppose that the structural material is isotropic, with elastic modulus E and Poisson's ratio ν. The in-plane stress-strain relations for plane stress and plane strain as given in any textbook on elasticity are

$$\textit{plane stress:}\quad \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ 2\epsilon_{xy}\end{bmatrix}$$

$$\textit{plane strain:}\quad \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ 2\epsilon_{xy}\end{bmatrix}$$

a) Show that the constitutive matrix of plane strain can be formally obtained by replacing E by a fictitious modulus E^* and ν by a fictitious Poisson's ratio ν^* in the plane stress constitutive matrix. Find the expression of E^* and ν^* in terms of E and ν.

b) Do also the inverse process: go from plane strain to plain stress by replacing a fictitious modulus and Poisson's ratio in the plane strain constitutive matrix.

To be able to find this two new expression for E and ν, in each case the same stress component of the stresses from plane stress and strain is going to be used, in this case σ_{xx}. From them, the members that multiply ε_{xx} and ε_{yy} will be contrasted in order to obtain the new expressions. Then it will be checked that using the new variables E^* and ν^*, plain strain can be obtained from plain stress and the opposite in each case.

(Solutions attached in next pages)

P. Strain: $\sigma_{xx} = \underbrace{\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}} \varepsilon_{xx} + \underbrace{\dfrac{E\nu}{(1+\nu)(1-2\nu)}} \varepsilon_{yy}$

P. Stress: $\sigma_{xx} = \underbrace{\dfrac{E}{1-\nu^2}} \varepsilon_{xx} + \underbrace{\dfrac{E\nu}{1-\nu^2}} \varepsilon_{yy}$

- Sustituying E^* and ν^* in P. Stress

$$\left.\begin{aligned} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} &= \frac{E^*}{1-\nu^{*2}} \\ \frac{E\nu}{(1+\nu)(1-2\nu)} &= \frac{E^*\nu^*}{(1-\nu^{*2})} \end{aligned}\right\}$$

① $E = \dfrac{E^*(1+\nu)(1-2\nu)}{(1-\nu)} \dfrac{1}{(1-\nu^{*2})} \longrightarrow$ Sust. in the other equation

② $\dfrac{\cancel{E^*}\cancel{(1+\nu)}\cancel{(1-2\nu)}}{(1-\nu)\cancel{(1-\nu^{*2})}} \dfrac{\nu}{\cancel{(1+\nu)}\cancel{(1-2\nu)}} = \dfrac{\cancel{E^*}\nu^*}{\cancel{(1-\nu^{*2})}}$

$$\boxed{\nu^* = \frac{\nu}{1-\nu}} \longrightarrow \boxed{\nu = \nu^*(1-\nu)}$$

- Now, replacing the new expression for ν in ①

$$E = \frac{E^*(1+\nu)(1-2\nu)}{(1-\nu)(1-\nu^{*2})} = \frac{E^*(1+\nu)(1-2\nu)}{1-\nu} \frac{1}{1-\left(\frac{\nu}{1-\nu}\right)^2} =$$

As $(1-a)(1+a) = 1-a^2$

$$= \frac{E^*(1+\nu)(1-2\nu)}{1-\nu} \frac{1}{\left(1-\frac{\nu}{1-\nu}\right)\left(1+\frac{\nu}{1-\nu}\right)} =$$

$$= \frac{E^*(1+\nu)\cancel{(1-2\nu)}}{1-\nu} \cdot \frac{(1-\nu)(1-\nu)}{\cancel{(1-2\nu)}} =$$

$$\left.\begin{array}{l} = \dfrac{E^*(1+\nu)}{1-\nu}(1-\nu)^2 \\[2ex] \text{As } (1+\nu) = \dfrac{1-\nu^2}{1-\nu} \end{array}\right\} \quad E = E^* \frac{(1-\nu^2)}{(1-\nu)^2}(1-\nu^2) = E^*(1-\nu^2)$$

$$E = E^*(1-\nu^2) \longrightarrow \boxed{E^* = \frac{E}{(1-\nu^2)}}$$

- Now we prove that using E^* and ν^* in P. Stress matrix we can achieve the P. Strain one.

- Taking as example σ_{xx}

$$\sigma_{xx} = \frac{E^*}{1-\nu^{*2}}\varepsilon_{xx} + \frac{E^*\nu^*}{1-\nu^{*2}}\varepsilon_{yy}$$

1 $$\frac{E^*}{1-\nu^{*2}} = \frac{E/(1-\nu^2)}{1-\left(\frac{\nu}{1-\nu}\right)^2} = \frac{E/(1-\nu^2)}{\left(\frac{1}{1-\nu}\right)\left(\frac{1-2\nu}{1-\nu}\right)} = \frac{E(1-\nu)^2}{(1-\nu^2)(1-2\nu)} =$$

$$= \frac{E(1-\nu)^{\cancel{2}}}{\cancel{(1-\nu)}(1+\nu)(1-2\nu)} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

2 $$\frac{E^*\nu^*}{1-\nu^{*2}} = \frac{E/(1-\nu^2)}{1-\left(\frac{\nu}{1-\nu}\right)^2}\left(\frac{\nu}{1-\nu}\right) = \frac{E\cancel{(1-\nu)}}{(1+\nu)(1-2\nu)}\frac{\nu}{\cancel{(1-\nu)}} = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\sigma_{xx} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{xx} + \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{yy} \rightarrow \text{PLAIN STRAIN}$$

- Sustituying in P. Strain $\in^*$ and ν^*

$$\left.\begin{aligned} \frac{\in}{1-\nu^2} &= \frac{\in^*(1-\nu^*)}{(1+\nu^*)(1-2\nu^*)} \\ \frac{\in\nu}{(1-\nu^2)} &= \frac{\in^*\nu^*}{(1+\nu^*)(1-2\nu^*)} \end{aligned}\right\}$$

① $\in = \dfrac{\in^*(1-\nu^*)(1-\nu^2)}{(1+\nu^*)(1-2\nu^*)}$ → Sustituying in the other equation

② $\dfrac{\cancel{\in}^*(1-\nu^*)\cancel{(1-\nu^2)}}{\cancel{(1+\nu^*)}\cancel{(1-2\nu^*)}} \dfrac{\nu}{\cancel{(1-\nu^2)}} = \dfrac{\cancel{\in}^*\nu^*}{\cancel{(1+\nu^*)}\cancel{(1-2\nu^*)}}$

$\nu = \dfrac{\nu^*}{1-\nu^*}$; $\nu - \nu^*\nu = \nu^*$

$\nu = \nu^* + (\nu\nu^*)$

$$\boxed{\frac{\nu}{1+\nu} = \nu^*}$$

- Now replacing in ① the ν value

$$\in = \frac{\in^*(1-\nu^*)\left(1-\left(\frac{\nu^*}{1-\nu^*}\right)^2\right)}{(1+\nu^*)(1-2\nu^*)} = \frac{\in^*(1-\nu^*)\left(\frac{1}{1-\nu^*}\right)\left(\frac{1-2\nu^*}{1-\nu^*}\right)}{(1+\nu^*)(1-2\nu^*)} =$$

$$= \frac{\in^*\frac{(1-2\nu^*)}{(1-\nu^*)}}{(1+\nu^*)(1-2\nu^*)} = \frac{\in^*}{(1+\nu^*)(1-\nu^*)} = \frac{\in^*}{(1-\nu^{*2})}$$

$$\left\{\begin{aligned} \in^* &= \in(1-\nu^{*2}) \\ \in^* &= \in\left(1-\left(\frac{\nu}{1+\nu}\right)^2\right) = \frac{1+2\nu}{(1+\nu)^2} \end{aligned}\right.$$

- Then it can be prove that using E^* and ν^* in P. Strain, P. Stress can be achieved.

$$\sigma_{xx} = \frac{E^*(1-\nu^*)}{(1+\nu^*)(1-2\nu^*)}\varepsilon_{xx} + \frac{E^*\nu^*}{(1+\nu^*)(1-2\nu^*)}\varepsilon_{yy}$$

1 $$\frac{E^*(1-\nu^*)}{(1+\nu^*)(1-2\nu^*)} = \frac{E(1-\nu^{*2})(1-\nu^*)}{(1+\nu^*)(1-2\nu^*)} =$$

$$= \frac{E(1-\nu^*)\cancel{(1+\nu^*)}(1-\nu^*)}{\cancel{(1+\nu^*)}(1-2\nu^*)} = \frac{E(1-\nu^*)^2}{1-2\nu^*} =$$

$$= \frac{E\left(1-\frac{\nu}{\nu+1}\right)^2}{1-2\left(\frac{\nu}{1+\nu}\right)} = \frac{E\left(\frac{1+\nu-\nu}{\nu+1}\right)^2}{\frac{1+\nu-2\nu}{\nu+1}} =$$

$$= \frac{E(\nu+1)}{(\nu+1)^2(1-\nu)} = \frac{E}{1-\nu^2}$$

2 $$\frac{E^*\nu^*}{(1+\nu^*)(1-2\nu^*)} = \frac{E(1-\nu^{*2})\nu^*}{(1+\nu^*)(1-2\nu^*)} = \frac{E(1-\nu^*)(1+\nu^*)\nu^*}{(1+\nu^*)(1-2\nu^*)} =$$

$$= \frac{E(1-\nu^*)\nu^*}{(1-2\nu^*)} = \frac{E\left(1-\frac{\nu}{1+\nu}\right)\frac{\nu}{1+\nu}}{1-\frac{2\nu}{1+\nu}} = \frac{E\left(1-\frac{\nu}{1+\nu}\right)\frac{\nu}{1+\nu}}{\frac{1-\nu}{1+\nu}} =$$

$$= \frac{E\nu(1+\nu)}{(1+\nu)^2(1-\nu)} = \frac{E\nu}{1-\nu^2}$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx} + \frac{E\nu}{1-\nu^2}\varepsilon_{yy} \rightarrow \text{PLAIN STRESS}$$

Assignment 3.2

In the finite element formulation of near incompressible isotropic materials, it is convenient to use the so-called Lamé constants λ and μ instead of E and ν in the constitutive equations. Both λ and

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \mu = G = \frac{E}{2(1+\nu)}$$

μ have the physical dimension of stress and are related to E and ν by

a) Find the inverse relations for E and ν in terms of λ and μ.

   Using the expressions above, new expression for E and ν will be obtained in terms of λ and μ.

b) Express the elastic matrix for plane stress and plane strain cases in terms of λ and μ.
   Substituting the expressions obtained in point a for E and ν in the elastic matrix (see assignment 3.1), new expressions in terms of λ and μ are obtained.

   For the plane strain matrix, it will be obtained a simpler expression due to the fact that λ/ν is equal to the value multiplying outside the hole matrix.

c) Split the stress-strain matrix E of plane strain as $\boldsymbol{E}=\boldsymbol{E}\lambda+\boldsymbol{E}\mu$ in which $\boldsymbol{E}\lambda$ and $\boldsymbol{E}\mu$ contain only λ and μ respectively. This is the Lamé splitting of the plane strain constitutive equations, which leads to the so-called B.bar formulation of near-incompressible finite elements.

   From the obtained elastic matrix for plane strain it is not difficult to divide the matrix in two matrices according with the requests of the exercise.

   This new E matrix definition can be seen as a deviatoric + volumetric splitting.

d) Express E_λ and E_μ also in terms of E and ν.
   Using the given expressions at the beginning of the exercise, an replacing λ and μ with them, the new way to expres E_λ and E_μ is obtained.

(Solutions attached in next pages)

a)

- From the given expressions
$$\begin{cases} E = \dfrac{\lambda(1+\nu)(1-2\nu)}{\nu} \\[2ex] \nu = \dfrac{E}{2\mu} - 1 \end{cases}$$

- Sustituying ν in E:

$$E = \frac{\lambda\left(1+\frac{E}{2\mu}-1\right)\left(1-2\left(\frac{E}{2\mu}-1\right)\right)}{\frac{E}{2\mu}-1} = \frac{\lambda\left(\frac{E}{2\mu}\right)\left(3-\frac{E}{\mu}\right)}{\frac{E}{2\mu}-1} =$$

$$= \frac{\lambda\left(\frac{E}{2\mu}\right)\left(3-\frac{E}{\mu}\right)}{\frac{E}{2\mu}-\frac{2\mu}{2\mu}} = \frac{\lambda\cancel{E}\left(3-\frac{E}{\mu}\right)}{E-2\mu} = \cancel{E}$$

$$1 = \frac{\lambda\left(3-\frac{E}{\mu}\right)}{E-2\mu} \; ; \qquad E - 2\mu = \lambda 3 - \frac{\lambda E}{\mu}$$

$$E + \frac{\lambda}{\mu}E = 3\lambda + 2\mu$$

$$E = \frac{3\lambda+2\mu}{\left(1+\frac{\lambda}{\mu}\right)} = \frac{3\lambda+2\mu}{\left(\frac{\mu+\lambda}{\mu}\right)}$$

$$\boxed{E = \mu\left(\frac{3\lambda+2\mu}{\mu+\lambda}\right)}$$

- Now ν is calculated

$$\nu = \frac{3\lambda+2\mu}{2(\mu+\lambda)} - 1 = \frac{3\lambda+\cancel{2\mu}-\cancel{2\mu}-2\lambda}{2(\mu+\lambda)} = \frac{\lambda}{2(\mu+\lambda)}$$

$$\boxed{\nu = \frac{\lambda}{2(\mu+\lambda)}}$$

b) & c)

- STRAIN MATRIX $\dfrac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$

- As $\dfrac{\lambda}{\nu} = \dfrac{E}{(1+\nu)(1-2\nu)}$, we obtain

$$\frac{\lambda}{\nu}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} = \lambda \begin{bmatrix} \frac{1}{\nu}-1 & 1 & 0 \\ 1 & \frac{1}{\nu}-1 & 0 \\ 0 & 0 & \frac{1}{2\nu}-1 \end{bmatrix} =$$

① $\dfrac{1}{\nu}-1 = \dfrac{2(\lambda+\mu)}{\lambda} - 1 = \dfrac{2\lambda+2\mu-\lambda}{\lambda} = \dfrac{\lambda+2\mu}{\lambda}$

② $\dfrac{1}{2\nu}-1 = \dfrac{\lambda+\mu}{\lambda} - 1 = \dfrac{\lambda+\mu-\lambda}{\lambda} = \dfrac{\mu}{\lambda}$

$$= \lambda \begin{bmatrix} \frac{\lambda+2\mu}{\lambda} & 1 & 0 \\ 1 & \frac{\lambda+2\mu}{\lambda} & 0 \\ 0 & 0 & \frac{\mu}{\lambda} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & 1/\lambda & 0 \\ 1/\lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

- Now, the matrix E can be written as $E = E_\mu + E_\lambda$

$$[E] = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} + \begin{bmatrix} \lambda & 1/\lambda & 0 \\ 1/\lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- STRESS MATRIX $\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

$$[E] = \frac{\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}}{1-\left(\frac{\lambda}{2(\lambda+\mu)}\right)^2}\begin{bmatrix} 1 & \frac{\lambda}{2(\lambda+\mu)} & 0 \\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0 \\ 0 & 0 & \frac{1}{2}-\frac{\lambda}{4(\lambda+\mu)} \end{bmatrix}$$

d)

- Taking E_μ and E_λ from above and λ and μ definitions

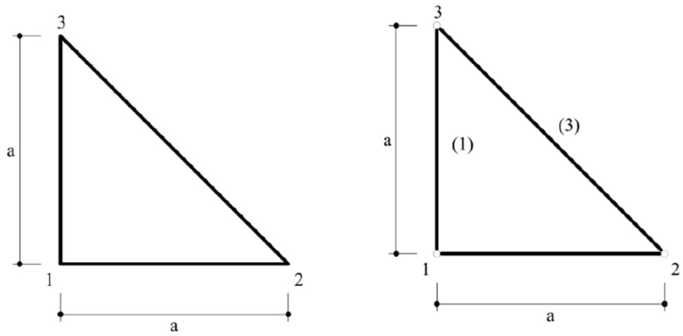
$$[E_\mu] = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} = \begin{bmatrix} \frac{E}{1+\nu} & 0 & 0 \\ 0 & \frac{E}{1+\nu} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix}$$

$$[E_\lambda] = \begin{bmatrix} \lambda & 1/\lambda & 0 \\ 1/\lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{(1+\nu)(1-2\nu)}{E\nu} & 0 \\ \frac{(1+\nu)(1-2\nu)}{E\nu} & \frac{E\nu}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Assignment 3.3

Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Let's consider for simplicity $a = h = 1$. The material parameters are E,ν. Initially ν _is set to zero. Two structural models are considered for this problem as depicted in the figure:

- A plane linear Turner triangle with the same dimension.
- A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are A1 = A2 and A3.



a) Calculate the stiffness matrix ***Ke*** for both models.
The Turner triangle is a three-node triangle with linear shape functions in which the degrees of freedom are located at the corners. The element stiffness matrix is calculated using the plane stress problem general formula, so that

$$K^e = B^T E B \int_{\Omega^e} h d\Omega$$

where h, the thickness is uniform over all the element.
For the second case, the stiffness matrix calculation introduced in the Assignment 1, will be use in order to obtain K^e for linear elements, in these case three bars.

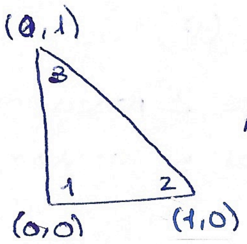
b) Is there any set of values for cross sections A1=A2=A and A3=A' to make both stiffness matrix equivalent: ***Kbar = Ktriangle***? If not, which are these values to make them as similar as possible?

c) Why these two stiffness matrix are not equivalent? Find a physical explanation.

d) Solve question a) considering ν≠0 and extract some conclusions.

(Solutions attached in next pages)

a) Stiffnes matrix k^e



1) Linear Turner triangle

$$\left.\begin{array}{l} a = h = 1 \\ E,\ \nu = 0 \end{array}\right\}$$

$$A = \frac{1}{2}$$

- If the thickness is assume to be constant (h)

$$k^e = \frac{h}{A4} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{32} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

where $\left.\begin{array}{l} x_{jk} = x_j - x_k \\ y_{jk} = y_j - y_k \end{array}\right\}$ coordinates of the nodes

$$[E] = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}$$

The constitutive matrix E, will assume to be constant over the element

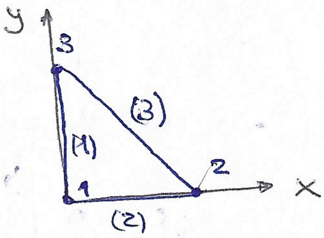
- As initially the $\nu = 0$, $[E] = E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$

- Substituting all the values into the matrix

$$k^e = \begin{bmatrix} 1{,}5 & 0{,}5 & -1 & -0{,}5 & -0{,}5 & 0 \\ 0{,}5 & 1{,}5 & 0 & -0{,}5 & -0{,}5 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -0{,}5 & -0{,}5 & 0 & 0{,}5 & 0{,}5 & 0 \\ -0{,}5 & -0{,}5 & 0 & 0{,}5 & 0{,}5 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \frac{E}{2}$$

2) Three bar elements ($\nu = 0$)

- Now we have three 2-noded elements in a 2D system, so that the global k matrix for each element (4x4) is defined as:



$$k^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

where $c = \cos(\alpha)$ and $s = \sin(\alpha)$, being α the angle formed by the bar and the global coordinates system

- Element 1 ($\alpha = 90^\circ$)

$$k^1 = EA_1 \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right]$$

- Element 2 ($\alpha = 0^\circ$)

$$k^2 = EA_2 \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- Element 3 ($\alpha = 90+45 = 135^\circ$)

$\alpha = 135^\circ$ — $\sin(135) = \sqrt{2}/2$, $\cos(135) = -\sqrt{2}/2$

$L_3 = \sqrt{1^2+1^2} = \sqrt{2}$

$$k^3 = \frac{EA_3}{\sqrt{2}} \left[\begin{array}{cc|cc} 1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ \hline -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{array}\right]$$

- Finally by the assembling process

$$K = \left[\begin{array}{cc|cc|cc} A_2 & 0 & -A_2 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & -A_1 \\ \hline -A_2 & 0 & A_2 + \frac{A_3}{2\sqrt{2}} & \frac{-A_3}{2\sqrt{2}} & \frac{-A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \\ 0 & 0 & \frac{-A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{-A_3}{2\sqrt{2}} \\ \hline 0 & 0 & \frac{-A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{-A_3}{2\sqrt{2}} \\ 0 & -A_1 & \frac{A_3}{2\sqrt{2}} & \frac{-A_3}{2\sqrt{2}} & \frac{-A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \end{array}\right] E$$

b)

- Looking to both matrixes, we can conclude that they will never be equal due to the fact that some k_{ij} elements in the bar case are equal to zero, whereas they are different from zero in the triangle case and vice versa. For example:

BAR	TRIANGLE
$k_{12} = 0$	$k_{12} = 0'5$
$k_{56} = \frac{-A_3}{\sqrt{2}}$	$k_{56} = 0$

- Then, we have to look for that values that will be our matrices as similar as possible.

- First of all, it is going to be checked what values k_{ij} are different from zero in both case. These values will be the ones that can be compared in order to obtain some results.

$$\begin{bmatrix} k_{11} & 0 & k_{13} & 0 & 0 & 0 \\ 0 & k_{22} & 0 & 0 & 0 & k_{26} \\ k_{31} & 0 & k_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{44} & k_{45} & 0 \\ 0 & 0 & 0 & k_{54} & k_{55} & 0 \\ 0 & k_{62} & 0 & 0 & 0 & k_{66} \end{bmatrix}$$

* $A_1 = A_2 = A$

- Looking the non zero k_{ij} elements of the matrices, the different values are obtained for A in order to achieve the same value of the elements.

- At k_{11}, k_{22}: $A = 1'5 \frac{1}{2} = 0'75$

- At k_{13}, k_{26}: $A = 1 \frac{1}{2} = 0'5$

- Then, for this first value A, in order to try to obtain the most similar numbers in all the kij elements involved, and as the two possible options are not so far, an average of both is going to be used.

$$\boxed{A = \frac{0'5 + 0'75}{2} = 0'625}$$

* $A_3 = A'$

- Looking to the non zero kij elements, three possible values are going to be obtained for A', one of them affected by the previous obtained A value.

- At k_{33}: $A + \frac{A'}{2\sqrt{2}} = 1\frac{1}{2} \rightarrow A' = \frac{-\sqrt{2}}{2} \rightarrow$ NOT POSSIBLE

- At k_{44}, k_{45}, k_{55}: $\frac{A'}{2\sqrt{2}} = 0'5\frac{1}{2} \rightarrow A' = \frac{\sqrt{2}}{2}$

- At k_{66}: $\frac{A'}{2\sqrt{2}} = 1\frac{1}{2} \rightarrow A' = \sqrt{2}$

- In this case, taking into account that the first value obtained is not feasible and that the 2nd value allow us to achieve four equal values between the matrices, it is the one that is going to be used.

$$\boxed{A' = \frac{\sqrt{2}}{2}}$$

- In that way, the other two kij affected elements will be no equal between that but they are not going to be so different either.

c) Why these two stiffness matrix are not equivalent? Find a physical explanation.

As it is known, in both cases the **K** stiffness matrix establish a relationship between the forces and the displacements at the nodes of the elements ($Ku = f$) that as it had been shown, it is not equal in both cases.

First of all talking about the bar structure in 2D, the bar is the 2-node simplest finite element that can be characterized by two properties: one preferred dimension, much more larger than the other two (in this case the longitudinal dimension); and the fact that it resist an internal axial force along its longitudinal dimension. So that, each member of the truss, whose properties are uniform along the length, can be interpreted as a linear spring of stiffness $k = \frac{AE}{L}$. The stiffness matrix is computed for each bar element to later being assembled following the way they are joined in the total structure.

Now, the Turner triangle is introduced in order to obtain the main differences between the cases. The Turner triangle is a 3-node element with linear shape functions that represent the simplest triangular element for the plane stress problem, that will no longer support just axial forces along its length, but two directional loads. To be able to introduce this new condition to the problem, new elements as the shape function or the strain-displacement matrices have to be added to the problem, as the constitutive matrix E in order to interpreted the behavior of the material over the element.

Now, it is easily to understand that, such a different elements, 2-node with just axial loads and 3-node material triangle in which the element support loads in both direction of the space can't be interpreted as the same problem.

d)

- If $\nu \neq 0$ the k Turner triangle stiffness matrix will change due to the fact that it depend on the E constitutive matrix (plane stress problem).
- The new k^e matrix is

$$k^e = \frac{E}{2(1-\nu^2)} \begin{bmatrix} 3/2 - \nu/2 & \nu/2 + 1/2 & -1 & \nu/2 - 1/2 & \nu/2 - 1/2 & -\nu \\ \nu/2 + 1/2 & 3/2 - \nu/2 & -\nu & \nu/2 - 1/2 & \nu/2 - 1/2 & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ \nu/2 - 1/2 & \nu/2 - 1/2 & 0 & 1/2 - \nu/2 & 1/2 - \nu/2 & 0 \\ \nu/2 - 1/2 & \nu/2 - 1/2 & 0 & 1/2 - \nu/2 & 1/2 - \nu/2 & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

- The main difference is that now, some k_{ij} elements stop to be equal to zero. When $\nu = 0$, looking the plane stress E matrix, it is observed that

$$\sigma_{xx} = E\,\varepsilon_{xx}$$
$$\sigma_{yy} = E\,\varepsilon_{yy}$$
$$\sigma_{xy} = \frac{1}{2} 2 E_{xy} = \varepsilon_{xy}$$

so that, the principal stresses just depend on their direction strain. However if $\nu \neq 0$,

$$\sigma_{xx} = f(\varepsilon_{xx}, \varepsilon_{yy})$$
$$\sigma_{yy} = f(\varepsilon_{xx}, \varepsilon_{yy})$$

Now, the stresses suffered by the material in each direction depend on the other direction strain too. So the Poisson parameter adds the fenomenon that the material try to modify its length in the direction perpendicular to the load/stress direction.

- If $\nu \neq 0$, as the bar element just support loads in the axial direction will be not affected the k^e matrix.