As3-Benjaminsson:

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3.1:

Given:
Plane stress:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ 2\epsilon_{xy}\end{bmatrix} \qquad (1.1)$$

Plane strain:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ 2\epsilon_{xy}\end{bmatrix} \qquad (1.2)$$

Solution:
a)
By using the fictious variables $E = E^*, \nu = \nu^*$ the following system is recieved from (1.1)

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E^*}{1-\nu^{*2}}\begin{bmatrix}1 & \nu^* & 0\\ \nu^* & 1 & 0\\ 0 & 0 & \frac{1-\nu^*}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ 2\epsilon_{xy}\end{bmatrix}. \qquad (1.3)$$

To also hold the requirements for plane strain, each element in (1.3) must match the corresponding element in (1.2). So by setting equal matrix element A_{22} of (1.3) to A_{22} of (1.2) it follows

$$\frac{E^*}{1-\nu^{*2}} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$E^* = \frac{E(1-\nu)(1-\nu^{*2})}{(1+\nu)(1-2\nu)}. \qquad (1.4)$$

Further setting A_{33} in (1.3) equal to A_{33} in (1.2) it follows

$$\frac{E^*}{2(1+\nu^*)} = \frac{E}{2(1+\nu)}$$

$$E^* = \frac{E(1+\nu^*)}{(1+\nu)}. \qquad (1.5)$$

Substituting the value of E^* from equation (1.5) into (1.4) gives

$$\frac{E(1+\nu^*)}{(1+\nu)} = \frac{E(1-\nu)(1-\nu^{*2})}{(1+\nu)(1-2\nu)}$$
$$\frac{1}{1-\nu^*} = \frac{(1-\nu)}{(1-2\nu)}$$
$$\nu^* = \frac{\nu}{1-\nu} \qquad (1.6)$$

Inserting ν^* from equation (1.6) into equation (1.5) yields

$$E^* = \frac{E}{1-\nu^2}. \qquad (1.7)$$

b)
The same procedure is done, starting this time from plain strain

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ 2\epsilon_{xy}\end{bmatrix}.$$

Replacing with fictious modulus E^* and Poissons ratio ν^* the equation becomes

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E^*}{(1+\nu^*)(1-2\nu^*)}\begin{bmatrix}1-\nu^* & \nu^* & 0\\ \nu^* & 1-\nu^* & 0\\ 0 & 0 & \frac{1-2\nu^*}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ 2\epsilon_{xy}\end{bmatrix}. \qquad (1.8)$$

For plane stress to be present, each element of the constitutive matrix in equation (1.8) must be equal to the corresponding element in (1.2).

For element A_{22} it must hold

$$\frac{E^*(1-\nu^*)}{(1+\nu^*)(1-2\nu^*)} = \frac{E^*(1-\nu^*)}{1-\nu^*-\nu^{*2}} = \frac{E^*(1-\nu^*)}{(1-\nu^*)(1+\nu^*)-\nu^*} = \frac{E^*}{1+\nu^*-\frac{\nu^*}{1-\nu^*}}$$
$$= \frac{E}{1-\nu^2} \qquad (1.9)$$

and for A_{33}

$$\frac{E^*\frac{1-2\nu^*}{2}}{(1+\nu^*)(1-2\nu^*)} = \frac{E}{1-\nu^2}\frac{1-\nu}{2}$$
$$\frac{E^*}{2(1+\nu^*)} = \frac{E}{2(1+\nu)}$$

$$E^* = E\frac{1+\nu^*}{1+\nu} \qquad (1.10)$$

Substituting E^* from equation (1.10) into equation (1.9) gives

$$\frac{E\frac{1+\nu^*}{1+\nu}(1-\nu^*)}{(1+\nu^*)(1-2\nu^*)} = \frac{E}{1-\nu^2}$$

$$\frac{\frac{1}{1+\nu}(1-\nu^*)}{(1-2\nu^*)} = \frac{1}{1-\nu^2}$$

$$\frac{(1-\nu^*)}{(1-2\nu^*)} = \frac{1}{1-\nu}$$

$$\frac{\nu^*}{1-\nu^*} = \nu$$

$$\nu^* = \frac{\nu}{1+\nu} \qquad (1.11).$$

Substituting ν^* from equation (1.11) into (1.10) gives

$$E^* = E\frac{1+2\nu}{(1+\nu)^2}. \qquad (1.12)$$

3.2:

Given:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad (2.1)$$

$$\mu = G = \frac{E}{2(1+\nu)} \qquad (2.2)$$

Solution:

a)

Using equation (2.1) and equation (2.2) the term E can be isolated as follows

$$E = \frac{\lambda(1+\nu)(1-2\nu)}{\nu}$$

$$E = 2(1+\nu)\mu \qquad (2.3)$$

giving

$$\frac{\lambda(1+\nu)(1-2\nu)}{\nu} = 2(1+\nu)\mu$$

$$2\nu(\lambda+\mu) = \lambda$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)}. \qquad (2.4)$$

Substituting ν from equation (2.4) into equation (2.3) gives

$$E = \frac{3\lambda+2\mu}{\lambda+\mu}\mu \qquad (2.5)$$

b)
The elastic matrix for plane stress is

$$\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}.$$

Using equation (2.4) and equation (2.5) the matrix is rewritten as

$$\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{\frac{3\lambda+2\mu}{\lambda+\mu}\mu}{1-\left(\frac{\lambda}{2(\lambda+\mu)}\right)^2}\begin{bmatrix} 1 & \frac{\lambda}{2(\lambda+\mu)} & 0 \\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0 \\ 0 & 0 & \frac{1-\frac{\lambda}{2(\lambda+\mu)}}{2} \end{bmatrix} =$$

$$= \frac{\frac{3\lambda+2\mu}{\lambda+\mu}\mu}{\frac{4(\lambda+\mu)^2-\lambda^2}{4(\lambda+\mu)^2}}\begin{bmatrix} 1 & \frac{\lambda}{2(\lambda+\mu)} & 0 \\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0 \\ 0 & 0 & \frac{\lambda+2\mu}{4(\lambda+\mu)} \end{bmatrix} =$$

$$= \frac{(3\lambda+2\mu)4(\lambda+\mu)}{(2(\lambda+\mu)+\lambda)(2(\lambda+\mu)-\lambda)}\mu\begin{bmatrix} 1 & \frac{\lambda}{2(\lambda+\mu)} & 0 \\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0 \\ 0 & 0 & \frac{\lambda+2\mu}{4(\lambda+\mu)} \end{bmatrix} =$$

$$= \frac{4(\lambda+\mu)}{(\lambda+2\mu)}\mu\begin{bmatrix} 1 & \frac{\lambda}{2(\lambda+\mu)} & 0 \\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0 \\ 0 & 0 & \frac{\lambda+2\mu}{4(\lambda+\mu)} \end{bmatrix}. \qquad (2.6)$$

The elastic matrix for plane strain is

$$\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}.$$

Using (2.4) and (2.5) the elastic matrix is rewritten as

$$\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}=$$

$$=\frac{\frac{3\lambda+2\mu}{\lambda+\mu}\mu}{\left(1+\frac{\lambda}{2(\lambda+\mu)}\right)\left(1-2\frac{\lambda}{2(\lambda+\mu)}\right)}\begin{bmatrix}1-\frac{\lambda}{2(\lambda+\mu)} & \frac{\lambda}{2(\lambda+\mu)} & 0\\ \frac{\lambda}{2(\lambda+\mu)} & 1-\frac{\lambda}{2(\lambda+\mu)} & 0\\ 0 & 0 & \frac{1-2\left(\frac{\lambda}{2(\lambda+\mu)}\right)}{2}\end{bmatrix}=$$

$$=\frac{\frac{3\lambda+2\mu}{\lambda+\mu}\mu}{\left(\frac{3\lambda+2\mu}{2(\lambda+\mu)}\right)\left(\frac{\mu}{(\lambda+\mu)}\right)}\begin{bmatrix}\frac{\lambda+2\mu}{2(\lambda+\mu)} & \frac{\lambda}{2(\lambda+\mu)} & 0\\ \frac{\lambda}{2(\lambda+\mu)} & \frac{\lambda+2\mu}{2(\lambda+\mu)} & 0\\ 0 & 0 & \frac{\mu}{2(\lambda+\mu)}\end{bmatrix}$$

$$=2(\lambda+\mu)\begin{bmatrix}\frac{\lambda+2\mu}{2(\lambda+\mu)} & \frac{\lambda}{2(\lambda+\mu)} & 0\\ \frac{\lambda}{2(\lambda+\mu)} & \frac{\lambda+2\mu}{2(\lambda+\mu)} & 0\\ 0 & 0 & \frac{\mu}{2(\lambda+\mu)}\end{bmatrix}$$

$$=\begin{bmatrix}\lambda+2\mu & \lambda & 0\\ \lambda & \lambda+2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}\ (2.7)$$

c)
The elastic matrix can be split up as

$$E=\begin{bmatrix}\lambda+2\mu & \lambda & 0\\ \lambda & \lambda+2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}=E_\lambda+E_\mu$$

which gives

$$E_\lambda=\begin{bmatrix}\lambda & \lambda & 0\\ \lambda & \lambda & 0\\ 0 & 0 & 0\end{bmatrix}\qquad (2.8)$$

$$E_\mu=\begin{bmatrix}2\mu & 0 & 0\\ 0 & 2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}.\qquad (2.9)$$

d)
Remembering

$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)}\ (2.1)$$

$$\mu=\frac{E}{2(1+\nu)}\ (2.2),$$

it follows that

$$E_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} \qquad (2.10)$$

$$E_\mu = \frac{E}{2(1+\nu)}\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad (2.11).$$

3.3:

Given:

$$a = h = 1$$
$$\nu = 0$$

Solution:

a)

Global coordinates are set giving the nodes the following coordinates

$$1:\ x_1 = 0, y_1 = 0$$
$$2: x_2 = 1, y_2 = 0$$
$$3: x_3 = 0, y_3 = 1.$$

The area for the triangles are

$$2A = det = \begin{bmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{bmatrix} = 1$$
$$A = \frac{1}{2}.$$

The stress-strain matrix for plane stress is

$$\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix} = \{\nu = 0\} = E\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}.$$

Turner

The stiffness matrix for a Turner triangle is

$$K^e = \int_{\Omega^e} hB^T EBd\Omega = B^T EB\int_{\Omega^e} hd\Omega = \{B, E \text{ constant over } A, h \text{ constant}\} =$$
$$= \frac{h}{4A}\begin{bmatrix}y_{23} & 0 & x_{32}\\ 0 & x_{32} & y_{23}\\ y_{31} & 0 & x_{13}\\ 0 & x_{13} & y_{31}\\ y_{12} & 0 & x_{21}\\ 0 & x_{21} & y_{12}\end{bmatrix}\begin{bmatrix}E_{11} & E_{12} & E_{13}\\ E_{21} & E_{22} & E_{23}\\ E_{31} & E_{32} & E_{33}\end{bmatrix}\begin{bmatrix}y_{23} & 0 & y_{31} & 0 & y_{12} & 0\\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21}\\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12}\end{bmatrix},$$

where

$$x_{jk} = x_j - x_k$$

$$y_{jk} = y_j - y_k.$$

Using

$$1:\ x_1 = 0, y_1 = 0$$
$$2: x_2 = 1, y_2 = 0$$
$$3: x_3 = 0, y_3 = 1$$

the coordinate terms in the equation become

$$y_{23} = -1$$
$$x_{32} = -1$$
$$y_{31} = 1$$
$$x_{13} = 0$$
$$y_{12} = 0$$
$$x_{21} = 1$$

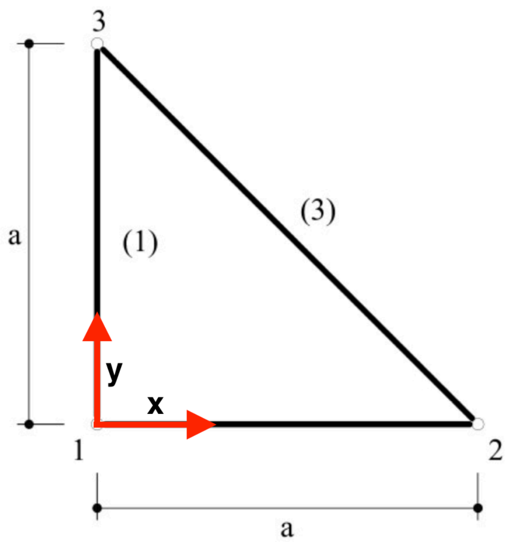
resulting in the final expression

$$K_{Turner} = \frac{Eh}{4A}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} = $$
$$= \frac{E}{2}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Bar elements

Given:

$$A_1 = A_2$$



Observing the problem as three members the element stiffness matrix is expressed

$$K^e = \frac{E^e A^e}{l^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix},$$

where

$$c = cos\phi, s = sin\phi$$

and ϕ is the angle between the local and global coordinate axis. We assume E is constant over the element.

For element (1) we have $\phi = \frac{\pi}{2}$, $l^{(1)} = a = 1$, and $A^{(1)} = A_1$ which gives

$$K^{(1)} = EA_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

For element (2) we have $\phi = 0$, $l^{(2)} = a = 1$, and $A^{(2)} = A_2 = A_1$ which gives

$$K^{(2)} = EA_1 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For element (3) we have $\phi = \frac{\pi}{4}$, $l^{(3)} = \sqrt{2}$, and $A^{(3)} = A_3$ which gives

$$K^{(3)} = \frac{EA_3}{\sqrt{2}} \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}.$$

Expanding the element matrices and assembling results in

$$K_{bars} = E\begin{bmatrix} A_1 & 0 & -A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & -A_1 \\ -A_1 & 0 & A_1 + \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & -A_1 & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & A_1 + \frac{A_3}{2\sqrt{2}} \end{bmatrix}.$$

b)
We have

$$K_{turner} = E\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$K_{bars} = E\begin{bmatrix} A_1 & 0 & -A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & -A_1 \\ -A_1 & 0 & A_1 + \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & -A_1 & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & A_1 + \frac{A_3}{2\sqrt{2}} \end{bmatrix}$$

Comparing the matrices it is obvious that there are no values of A_1 and A_3 that will make all elements equal. Choosing $A_1 = \frac{3}{4}$ and $A_3 = \frac{1}{\sqrt{2}}$ will make most of the diagonal elements the same which is the best possible choice of areas.

c)
The stiffness matrices are not equal due to the fact that the Turner triangle is a 2D element while the bar members are 1D elements. Due to the fact that the bar member is 1D the element only offers resistance in one direction, parallell to its length. The Turner triangle is a 2D element and considers resistance in 2D between the nodes.

d)
With $\nu \neq 0$ the elastic matrix become

$$\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}.$$

and the Turner stiffness matrix becomes

$$K_{Turner} = \frac{Eh}{(1-\nu^2)4A}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} =$$

$$= \frac{Eh}{4A}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & -\nu & 1 & 0 & 0 & \nu \\ -\nu & -1 & \nu & 0 & 0 & 1 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \end{bmatrix} =$$

$$= \frac{E}{2}\begin{bmatrix} \frac{3-\nu}{2} & \frac{1+\nu}{2} & -1 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 \\ \frac{1+\nu}{2} & \frac{3-\nu}{2} & -\nu & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}.$$

The stiffness matrix for the three bar elements remains the same.