UPC - BARCELONA TECH
MSc Computational Mechanics
Spring 2018

Coputations Solid Mechanics & Dynamics

Assignment 3

Due 26/02/2018

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Assignment 3.1

On “The Plane Stress Problem”:

In isotropic elastic materials (as well as in plasticity and viscoelasticity) it is convenient to use the so-called Lamé constants λ and μ instead of E and ν in the constitutive equations. Both λ and μ have the physical dimension of stress and are related to E and ν by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \mu = G = \frac{E}{2(1+\nu)}$$

1. Find the inverse relations for E,ν in terms of λ,μ.
2. Express the elastic matrix for plane stress and plane strain cases in terms of λ,μ.
3. Split the stress-strain matrix E for plane strain as

$$\boldsymbol{E} = \boldsymbol{E}_{\boldsymbol{\lambda}} + \boldsymbol{E}_{\boldsymbol{\mu}}$$

in which Eμ and E_λ contain only μ and λ, respectively.

This is the Lamé {λ,μ} splitting of the plane strain constitutive equations, which leads to the so-called B-bar formulation of near-incompressible finite elements.

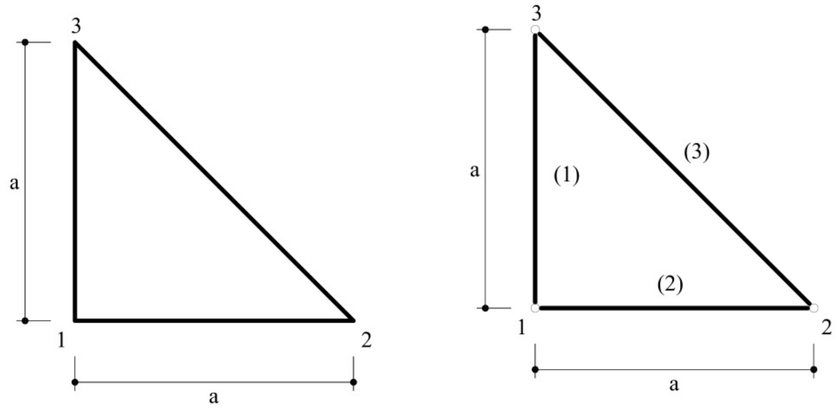
4. Express E_λ and E_μ also in terms of E and ν.

Assignment 3.2

On “The 3-node Plane Stress Triangle”:

Consider a plane triangular domain of thickness *h*, with horizontal and vertical edges of length *a*. Let us consider for simplicity $a = 1$, $h = 1$. The material parameters are E, ν. Initially ν is set to zero. Two discrete structural models are considered as depicted in the figure:

a) A plane linear Turner triangle with the same dimensions.
b) A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.



1. Calculate the stiffness matrices $\boldsymbol{K_{tri}}$ and $\boldsymbol{K_{bar}}$ for both discrete models.
2. Is there any set of values for the cross sections A_1=A_2 and A_3 to make both stiffness matrix equivalent: $\boldsymbol{K_{bar}} = \boldsymbol{K_{tri}}$? If not, which are the values that make them more similar?
3. Why these two stiffness matrices are not equal?. Find a physical explanation.
4. Consider nowidering $\nu \neq 0$ and extract some conclusions.

Date of Assignment: **19 / 02 / 2018**
Date of Submission: **26 / 02 / 2018**

The assignment must be submitted as a pdf file named **As3-Surname.pdf** to the CIMNE virtual center.

3.1 The problem on Plane Stress

Find the relations for E, ν in terms of λ, μ

Given that, $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = G = \dfrac{E}{2(1+\nu)}$. From the equation of μ, we get

$$E = 2\mu(1+\nu) \tag{1}$$

Substituting the above E in the equation of λ, we get

$$\lambda = \frac{2\nu\mu(1+\nu)}{(1+\nu)(1-2\nu)} = \frac{2\mu\nu}{(1-2\nu)}$$

$$\lambda(1-2\nu) = 2\mu\nu$$

$$\boxed{\nu = \frac{\lambda}{2(\lambda+\mu)}} \tag{2}$$

Substituting the value of E from equation (2) in equation (1),

$$E = 2\mu(1+\nu) = 2\mu\left(1 + \frac{\lambda}{2(\lambda+\mu)}\right) = 2\mu\left(\frac{3\lambda+2\mu}{2(\lambda+\mu)}\right)$$

$$\boxed{E = \frac{3\lambda+2\mu}{\lambda+\mu}} \tag{3}$$

Express Elastic matrix for plane stress and plane strain case in terms of λ, μ

The elastic matrix $\underline{E}$ for plane stress problem is given by

$$\underline{E}_{stress} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

After we simplify the matrix, we have three terms as follows and they are simplified further.

$$\frac{E}{(1-\nu^2)} = \frac{E}{(1-\nu)(1+\nu)} = \frac{2\mu}{(1-\nu)} = \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)}$$

$$\frac{E\nu}{(1-\nu^2)} = \frac{E\nu}{(1-\nu)(1+\nu2)} = \frac{2\mu\nu}{(1-\nu)} = \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)}.\frac{\lambda}{2(\lambda+\mu)} = \frac{2\mu\lambda}{(\lambda+2\mu)}$$

$$\frac{E}{(1-\nu^2)}.\frac{(1-\nu)}{2} = \frac{E}{(1-\nu)(1+\nu)}.\frac{(1-\nu)}{2} = \frac{E}{2(1+\nu)} = \mu$$

Thus, the elasticity matrix can be written in terms of λ and μ as follows,

$$\underline{E}_{stress} = \begin{bmatrix} \dfrac{4\mu(\lambda+\mu)}{(\lambda+2\mu)} & \dfrac{2\mu\lambda}{(\lambda+2\mu)} & 0 \\ \dfrac{2\mu\lambda}{(\lambda+2\mu)} & \dfrac{4\mu(\lambda+\mu)}{(\lambda+2\mu)} & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

The elastic matrix $\underline{\text{E}}$ for plane strain problem is given by

$$\underline{E}_{strain} = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \begin{bmatrix} 1 & \frac{\nu}{(1-\nu)} & 0 \\ \frac{\nu}{(1-\nu)} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

From the above matrix we have three terms to simplify

$$\frac{E(1-\nu)}{(1-2\nu)(1+\nu)} = \frac{\lambda(1-\nu)}{\nu} = 2\mu + \lambda$$
$$\frac{E\nu(1-\nu)}{(1-2\nu)(1+\nu)(1-\nu)} = \lambda$$
$$\frac{E(1-2\nu)(1-\nu)}{(1-2\nu)(1+\nu)(1-\nu)} = \frac{E}{2(1+\nu)} = \mu$$

Substituting all above values in the elasticity matrix, we have the following matrix:

$$\underline{E}_{strain} = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

Split the Stress-strain matrix $\underline{\textbf{E}}$ for plain strain case as
$\mathbf{E} = \mathbf{E}\lambda + \mathrm{E}\mu$

$$\underline{E}_{strain} = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

$$\underline{E}_{strain} = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

$$\underline{E}_{strain} = \lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Express $\mathbf{E}\lambda$ and $\mathbf{E}\mu$ in terms of E and ν
The λ and μ terms are replaced with the given equation in the question

$$\underline{E}_{\lambda} = \lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{E}_{\mu} = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.2 The 3-node Plane Stress Triangle

1 **Calculate the stiffness matrices** $K_{tri} and K_{bar}$

$x_2 = 0, x_2 = a, x_3 = 0, y_1 = 0, y_2 = 0, y_3 = a$

For Triangle $A = det\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = det\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} = \frac{a}{2} = \frac{1}{2}$

Plane stress elastic matrix,

$$\underline{E}_{stress} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

But, $\nu = 0$,

$$\therefore \underline{E}_{stress} = E\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

We know that $h = 1 =$ constant. $\therefore \underline{K}^e = \int_\Omega hB^T EBd\Omega$

Thus we can compute K^e directly as follows;

$$\underline{K}^e = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{21} & 0 & x_{13} \\ 0 & x_{32} & y_{23} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} .\underline{E}. \begin{bmatrix} y_{23} & 0 & y_{21} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{32} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{23} & x_{21} & y_{12} \end{bmatrix}$$

$x_{ij} = x_i - x_j; y_{ij} = y_i - y_j;$
$\therefore y_{23} = 0, x_{32} = -1, y_{21} = -1, x_{13} = 0, y_{12} = 1, x_{21} = 1$; After substituting and solving the above matrix we get

$$\boxed{\underline{K}_{tri} = \frac{E}{2}\begin{bmatrix} 1.5 & 0.5 & -1 & -0.5 & -0.5 & 0 \\ 0.5 & 1.5 & 0 & -0.5 & -0.5 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

For 3 bar structure, $E^{(1)}A^{(1)} = E^{(2)}A^{(2)} = EA^{(1)} and E^{(3)}A^{(3)} = EA^{(3)}$ The general stiffness matrix for any element can be written as follows

$$\underline{K}^{(e)} = \frac{E^{(e)}A^{(e)}}{L^{(e)}}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & c^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

Where, s = sinϕ, $c = cos\phi$ and ϕ is the angle between global and local axis

For Element 1,
$\phi = \pi/2, s = 1, c = 0$

$$\underline{K}^{(1)} = EA^{(1)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

For Element 2,
$\phi = 0, s = 0, c = 1$

$$\underline{K}^{(2)} = EA^{(1)} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For Element 3,
$\phi = 3\pi/4, s = \frac{\sqrt{2}}{2}, c = -\frac{\sqrt{2}}{2}$

$$\underline{K}^{(3)} = \frac{EA^{(3)}}{2\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The global stiffness matrix after the assembly is given as follows:

$$\boxed{\underline{K}_{bar} = E \begin{bmatrix} A_1 & 0 & -A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & -A_1 \\ -A_1 & 0 & A_1 + \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & -A_1 & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & A_1 + \frac{A_3}{2\sqrt{2}} \end{bmatrix}}$$

2 Is there any set of values for cross section $A_1 = A_2$ and A_3 to make both stiffness matrix equivalent $K_{bar} = K_{tri}$, if not, which are the values that make them more similar?

It is obvious to notice, its impossible to make $\underline{K}_{tri} = \underline{K}_{bar}$. In fact there are some zero coefficient entries in $\underline{K}_{bar}$ whereas the same entries in $\underline{K}_{tri}$ are different from zero. For example K_{14}, K_{15}

$$\underline{K}_{tri} = E \begin{bmatrix} 0.75 & 0.25 & -0.5 & -.250 & -0.25 & 0 \\ 0.25 & 0.75 & 0 & -0.25 & -0.25 & -0.5 \\ -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ -0.25 & -0.25 & 0 & 0.25 & 0.25 & 0 \\ -0.25 & -0.25 & 0 & 0.25 & 0.25 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

Now we can write $K_{tri,11} = K_{bar,11}$
$A_1 = 0.75, wonow K_{11} = K_{22} = 0.75, inbothmatrices.$
Also we get $K_{bar,13} = -0.75 \approx K_{tri,13} = -0.5$.
Now we could do $K_{55} = 0.5 = \frac{A_3}{2\sqrt{2}} \rightarrow A_3 = \sqrt{2}$

3 **Why these two stiffness matrix are not equal? Find a physical explanation.**

As discussed previously, both matrices are completely different. Let's consider now the physical interpretation of the coefficients of both $\underline{\mathrm{K}}$ matrices. The stiffness coefficients represent the force that would appear on a node upon prescribing a unit displacement on it and vanishing the other displacements. Here, we are analyzing a $3-$ bar structure and a triangular element.

In the case of the set of bar elements, we notice there is a node that is always not connected to one element. For instance, the node 3 is not a part of element 2, so we expect to have some zeros on the coefficients relating both clearly. When we try to make a stiffness coefficient equal in both matrices by closing certain values for A_1 and A_3 we are somewhat trying to reproduce the bar elements on the triangle, we can say without being really rigorous, that every point is connected to the element and there are zeros in those coefficients where $\nu = 0$. In other words, in the case of triangle the material is distributed throughout the whole element, whereas in the case of the bar, material properties are only considered where there is bar a bar.

Another fact that needs to be considered is that the $\underline{K}_{bar}$ matrix is derived from the truss model which is an abstraction of the reality considering just axial forces.

4 **Consider now $\nu \neq 0$ and extract some conclusions.**

If $\nu \neq 0$, then the matrix $\underline{\mathrm{E}}$ will be different. For this case, we would have

$$\underline{E}_{stress} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Now, this is still a symmetric matrix but no more a diagonal one. This basically means that stress in a particular direction are no more related. Only with the strain in that particular direction (the so called Poisson effect was neglected previously). Thus, we will now consider shear also.

Then we can state, in case of $\nu \neq 0$. Both models of analysis will be most different which translates into $\underline{K}_{bar}$ and $\underline{K}_{tri}$, which will be very dissimilar.