Master's Degree Numerical Methods in Engineering

UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

Computational Structural Mechanics and Dynamics

Assignment 3: Plane stress and linear triangle

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1 Stiffness matrix

On "Plane stress problem" and "Linear Triangle"

1. Compute the entries of K^e for the following plane stress triangle:

$$x_1 = 0, y_1 = 0, x_2 = 3, y_2 = 1, x_3 = 2, y_3 = 2$$

$$E = \begin{pmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{pmatrix}, h = 1$$

Partial result: $K_{11} = 18.75, K_{66} = 118.75$.

2. Show that the sum of the rows (and columns) 1, 3 and 5 of K_e as well as the sum of rows (and columns) 2, 4 and 6 must vanish, and explain why.

For computing the stiffness matrix of this triangular element we need to use the following formula:

$$K^e = h/4A \begin{pmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{pmatrix} \begin{pmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{pmatrix} \begin{pmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{pmatrix}$$

for which we have to calculate coordinate coefficients as:

$$x_{12} = -x_{21} = -3, y_{12} = -y_{12} = -1$$

$$x_{23} = -x_{32} = 1, y_{23} = -y_{32} = -1$$

$$x_{31} = -x_{13} = 2, y_{31} = -y_{13} = 2$$

Calculating the area of the triangle as:

$$2A = det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

So the are will be $A = 2$ and now we can calculate the stiffness matrix.

$$K^e = 1/8 \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{pmatrix}$$

$$K^e = 1/8 \begin{pmatrix} -100 & -25 & 50 \\ 25 & 100 & -50 \\ 200 & 50 & -100 \\ -50 & -200 & 100 \\ -100 & -25 & 150 \\ 75 & 300 & -50 \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 3 \\ 1 & -1 & -2 & 2 & 3 & -1 \end{pmatrix}$$

$$K^e = 1/8 \begin{pmatrix} 150 & 75 & -100 & -50 & -50 & -25 \\ 75 & 150 & 50 & 100 & -125 & -250 \\ -100 & 50 & 600 & -300 & -500 & 250 \\ -50 & 100 & -300 & 600 & 350 & -700 \\ -50 & -125 & -500 & 350 & 550 & -225 \\ -25 & -250 & 250 & -700 & -225 & 950 \end{pmatrix}$$

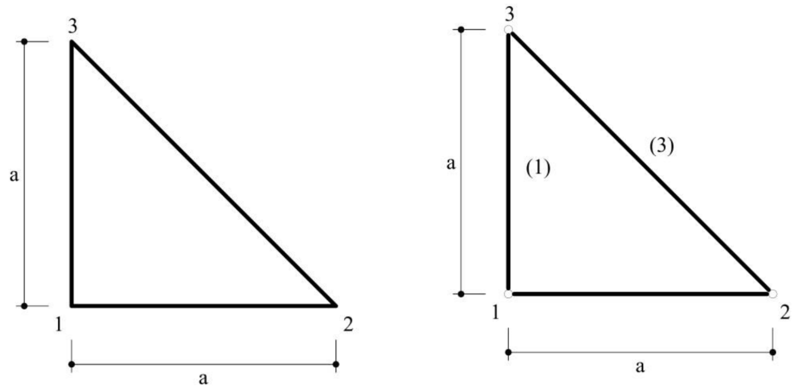
$$K^e = \begin{pmatrix} 18.7500 & 9.3750 & -12.5000 & -6.2500 & -6.2500 & -3.1250 \\ 9.3750 & 18.7500 & 6.2500 & 12.5000 & -15.6250 & -31.2500 \\ -12.5000 & 6.2500 & 75.0000 & -37.5000 & -62.5000 & 31.2500 \\ -6.2500 & 12.5000 & -37.5000 & 75.0000 & 43.7500 & -87.5000 \\ -6.2500 & -15.6250 & -62.5000 & 43.7500 & 68.7500 & -28.1250 \\ -3.1250 & -31.2500 & 31.2500 & -87.5000 & -28.1250 & 118.7500 \end{pmatrix}$$

The sum of all the rows and columns should be zero because the global K matrix should be singular so by applying the boundary conditions of the problem we can simplify the matrix and solve the equations and because the matrix is symmetric then the sum of the columns should be zero to. This property is not specific for this K matrix but is the property of every stiffness matrix as you can see in the following parts of the assignment for all the global stiffness matrices the sum of the rows and columns are zero.

2 2D elements and 1D bars

Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Let's consider for simplicity $a = h = 1$. The material parameters are E, ν. Initially ν is set to zero. Two structural models are considered for this problem as depicted in the figure:

A plane linear Turner triangle with the same dimensions.
A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are A1 = A2 and A3.



a) Calculate the stiffness matrix K^e for both models.

b) Is there any set of values for cross sections $A_1 = A_2 = A$ and $A_3 = A'$ to make both stiffness matrix equivalent: $K_{bar} = K_{triangle}$? If not, which are these values to make them as similar as possible?

c) Why these two stiffness matrix are not equivalent? Find a physical explanation.

d) Solve question a) considering $\nu \neq 0$ and extract some conclusions.

For the first case calculating the stiffness matrix is similar to the first part of the assignment, so we go through the same procedures(for the case of considering $\nu = 0$):

$$K = hE/4A \begin{pmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{pmatrix}$$

$$x_{12} = -x_{21} = -1, y_{12} = -y_{12} = 0$$

$$x_{23} = -x_{32} = 1, y_{23} = -y_{32} = -1$$

$$x_{31} = -x_{13} = 0, y_{31} = -y_{13} = 1$$

We calculate the K matrix using the values above, so we will have:

$$K = E/2 \begin{pmatrix} 1.5000 & 0.5000 & -1.0000 & -0.5000 & -0.5000 & 0 \\ 0.5000 & 1.5000 & 0 & -0.5000 & -0.5000 & -1.0000 \\ -1.0000 & 0 & 1.0000 & 0 & 0 & 0 \\ -0.5000 & -0.5000 & 0 & 0.5000 & 0.5000 & 0 \\ -0.5000 & -0.5000 & 0 & 0.5000 & 0.5000 & 0 \\ 0 & -1.0000 & 0 & 0 & 0 & 1.0000 \end{pmatrix}$$

For the second case we have to first calculate the stiffness matrix of each of the bar elements and then assemble them to have the global stiffness matrix. The stiffness matrix of a bar element is calculated as:

$$K^{(e)} = EA/L \begin{pmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{pmatrix}$$

so the element matrices will be:

$$K^{(2)} = EA_2 \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K^{(3)} = EA_3/\sqrt{2} \begin{pmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{pmatrix}$$

$$K^{(1)} = EA_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Considering $A_1 = A_2 = A_3$ and assembling the matrices we will have:

$$K = EA \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1+0.5/\sqrt{2} & -0.5/\sqrt{2} & -0.5/\sqrt{2} & 0.5/\sqrt{2} \\ 0 & 0 & -0.5/\sqrt{2} & 0.5/\sqrt{2} & 0.5/\sqrt{2} & -0.5/\sqrt{2} \\ 0 & 0 & -0.5/\sqrt{2} & 0.5/\sqrt{2} & 0.5/\sqrt{2} & -0.5/\sqrt{2} \\ 0 & -1 & 0.5/\sqrt{2} & -0.5/\sqrt{2} & -0.5/\sqrt{2} & 1+0.5/\sqrt{2} \end{pmatrix}$$

No it is not possible to have $K_{triangle} = k_{bar}$ only by having $A_1 = A_2 = A$ and $A_3 = A'$. In case we want them to be in the most similar situation is to consider $A' = A/\sqrt{2}$ in that case all the $\sqrt{2}$ will be gone from the bar k matrix and the matrices will be in the most similar situation.

The stiffness matrices can not be the same because the bar elements are only designed to stand axial forces, so it is only the combination of bar elements that makes a frame withstand different forces. The triangular element is a 2D element that has a surface so it is by nature able to withstand any 2D forces.

For the case where $\nu \neq 0$ the only difference will be in the E matrix and our K matrix will be calculated as:

$$K = hE/4A(1-\nu^2) \begin{pmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{pmatrix} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{pmatrix} \begin{pmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{pmatrix}$$

Calculating the k matrix with the values of part a. we will have:

$$K = hE/4A(1-\nu^2) \begin{pmatrix} 3/2-\nu/2 & \nu/2+1/2 & -1 & \nu/2-1/2 & \nu/2-1/2 & -\nu \\ \nu/2+1/2 & 3/2-\nu/2 & -\nu & \nu/2-1/2 & \nu/2-1/2 & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ \nu/2-1/2 & \nu/2-1/2 & 0 & 1/2-\nu/2 & 1/2-\nu/2 & 0 \\ \nu/2-1/2 & \nu/2-1/2 & 0 & 1/2-\nu/2 & 1/2-\nu/2 & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{pmatrix}$$

When we compare the case with $\nu \neq 0$ with the case that $\nu = 0$ we can see that basically the two stiffness matrices are the same but the one with the ν has some extra terms in the matrix related to ν and if we consider $\nu = 0$ for the case we will see that the stiffness matrix reduces to the first matrix that we have calculated. For the case of the 1D bars the stiffness matrix does not depend on the ν so considering $\nu \neq 0$ will not make any difference in the stiffness matrix calculated above.