**Universitat Politècnica de Catalunya**
Numerical Methods in Engineering
Computational Solid Mechanics and Dynamics

# Variational Formulation

Assignment 2

**Eduard Gómez**
February 22, 2020

## Contents

# 1 Assignment 2.1

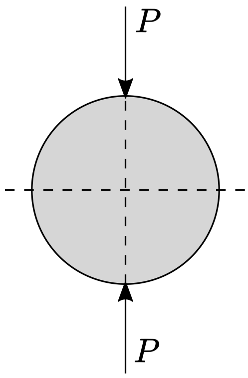
## 1.1 Statement

Identify the symmetry and antisymmetry lines.

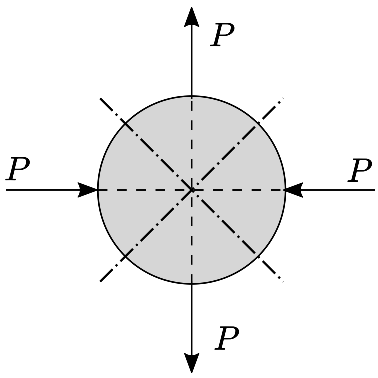
Then state whether it is possible to cut the complete structure to one half or one quarter before laying out a finite element mesh. Then draw a coarse FE mesh indicating, with rollers or fixed supports, which kind of displacement BCs you would specify on the symmetry or antisymmetry lines.

## 1.2 Solution

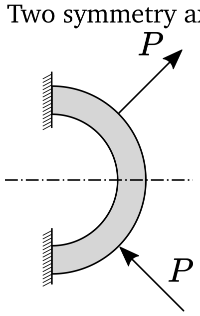
Here are the solutions for the first part of the statement. The symmetry axes are shown as dashed lines and the anti-symmetry axes as dash-dotted lines.



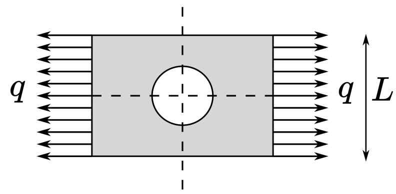

(a) Two symmetry axes



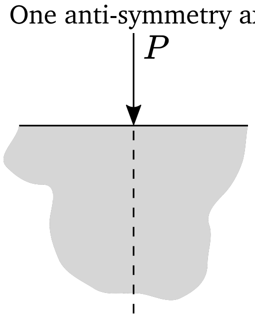

(b) Two symmetry and two anti-symmetry axes



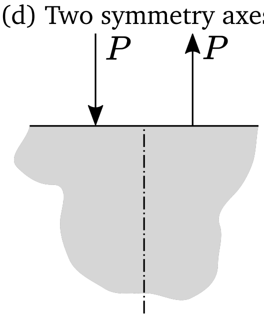

(c) One anti-symmetry axis




(d) Two symmetry axes




(e) One symmetry axis




(f) One anti-symmetry axis

Figure 1: Domains and their symmetries

Here are the solutions for the second part. Note that some rollers are connected to the mesh via a gray line. This offset is to make the figure clear, but acts the same as if the roller was put on the node it's connected to.

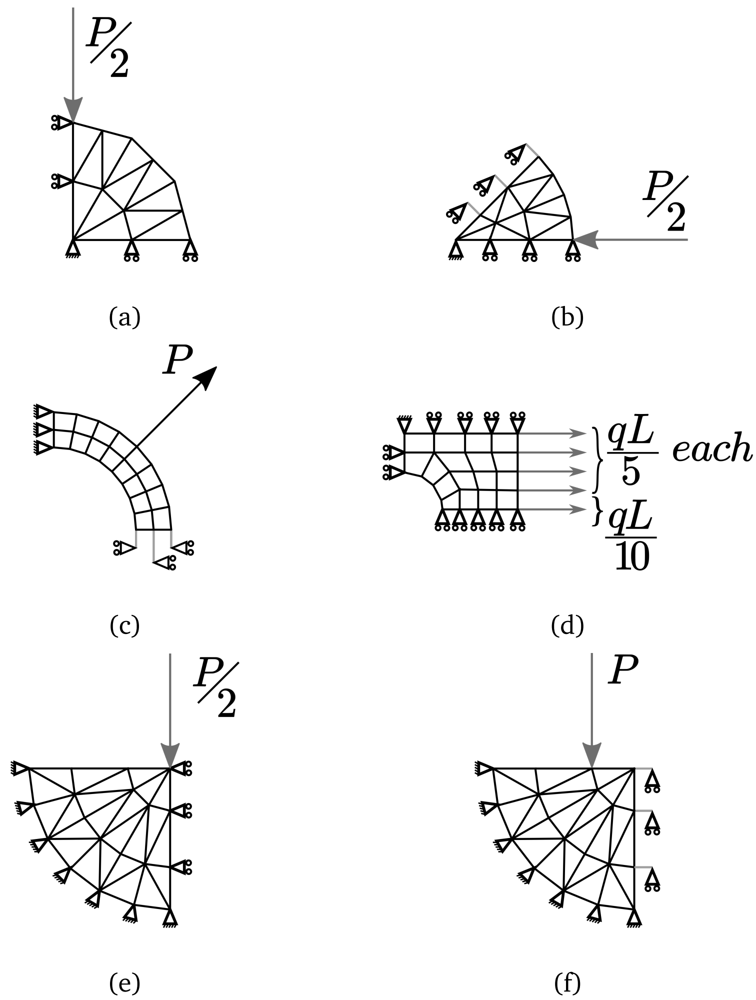



Figure 2: Meshed simplified domains with boundary conditions

## 1.3 Discussion

One can see how symmetries and anti-symmetries can always be exploited to reduce the size of the domain.

**Displacements** In symmetries, the displacements are restricted across the plane and free along it. In anti-symmetry, they are restricted along the plane and free to traverse it.

**Loads** Loads on the symmetry axis are split in half (e.g domain (a)). External loads cannot exist on the anti-symmetry axis so we need not worry about them.

# 2 Assignment 2.2

## 2.1 Statement

Explain the difference between "Verification" and "Validation" in the context of the FEM-Modelling procedure.

## 2.2 Solution

Validation is the process of evaluating the error that arises from simplifying reality down to a model. It stems from the assumptions taken when choosing the equations that describe the physical phenomenon, boundary conditions or domain.

In the context of FEM, the errors checked during validation are those committed on paper, before programming. These include but are not limited to choosing the wrong boundary conditions or the wrong constitutive laws.

Verification is the process of evaluating the error generated from the mathematical solver. It is therefore a purely mathematical concept. A solution might be perfectly verified but not be remotely close to the physical phenomenon it attempts to describe, as long as it satisfies the equations and conditions imposed by the model.

In the context of FEM, the errors checked during verification are those present in the code: assembly, solving the discretized system of equations, enforcing of boundary conditions, possible bugs, etc.

# 3 Assignment 2.3

## 3.1 Statement

A tapered bar element of length $h$ and areas $A_i$ and $A_j$ with $A$ interpolated as

$$A(\xi) = (1-\xi)A_i + \xi A_j \tag{1}$$

and constant density $\rho$ rotates on a plane at uniform angular velocity $\omega$ (rad/sec) about node $i$. Taking axis $x$ along the rotating bar with origin at node i, the centrifugal axial force is $q(x) = \rho A \omega^2 x$ along the length in which x is the longitudinal coordinate.

Find the consistent node forces as functions of $\rho$, $A_i$, $A_j$, $\omega$ and $h$, and specialize the result to the prismatic bar $A = A_i = A_j$.

## 3.2 Solution

Of course the force can be obtained as the integral of the distributed force along the entire domain:

$$N = \int_0^h q(x)\,dx = \int_0^h \rho A(\xi)\omega^2 x\,dx \tag{2}$$

We see that we are working in two diferent spaces. The geometric space $x \in [0, h]$ and the reference space $\xi \in [0, 1]$. We'll use the second one since it is more general. We need the following transformation:

$$x = h\xi \qquad dx = h\,d\xi$$

Applying it results in:

$$N = \rho\omega^2 \int_0^1 A(\xi)\xi h^2\,d\xi = \rho\omega^2 h^2 \int_0^1 \left[(1-\xi)A_1 + \xi A_2\right]\xi\,d\xi \tag{3}$$

Integrating yields the following expression:

$$N = \frac{\rho\omega^2 h^2}{6}(A_1 + 2A_2) \tag{4}$$

If we particularize for $A = A_1 = A_2$ we obtain:

$$N^* = \frac{1}{2}\rho A \omega^2 h^2 \tag{5}$$

If we consider all the mass of the bar to be at the center of mass $x_c = h/2$, then we obtain the high school formula for centrifugal force:

$$N^* = (\rho A h)\omega^2 \left(\frac{h}{2}\right) = m\omega^2 x_c \tag{6}$$

Thus giving the previous result more legitimacy.

If we, on the other hand, are not interested in the force at the nodes but rather the force vector, the development changes a bit. The integral now includes a shape function:

$$F_i = \int_0^h q(x)N_i(\xi)\,dx = \int_0^h \rho A(\xi)\omega^2 x N_i(\xi)\,dx \tag{7}$$

Applying the same coordinate transformation:

$$F_i = \int_0^h \rho A(\xi)\omega^2 h\xi N_i(\xi)\,dx = \rho\omega^2 h \int_0^h \left[(1-\xi)A_1 + \xi A_2\right]\xi N_i(\xi)\,dx \tag{8}$$

We'll use linear elements:

$$N_1(\xi) = 1 - \xi \qquad N_2(\xi) = \xi$$

Integrating equation 8 yields:

$$\left.\begin{aligned} F_1 &= \rho\omega^2 h \frac{A_1 h + A_2 h}{12} \\ F_2 &= \rho\omega^2 h \frac{A_1 h + 3A_2 h}{12} \end{aligned}\right\} \tag{9}$$

In vector form:

$$F = \frac{\rho\omega^2 h^2}{12}\begin{bmatrix} A_1 + A_2 \\ A_1 + 3A_2 \end{bmatrix} \tag{10}$$

Particularizing for $A = A_1 = A_2$:

$$F = \frac{\rho A\omega^2 h^2}{6}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{11}$$