**Computational Structural Mechanics and Dynamics**

# Assignment 10

Assignment: Dynamics

Oscar Salvador

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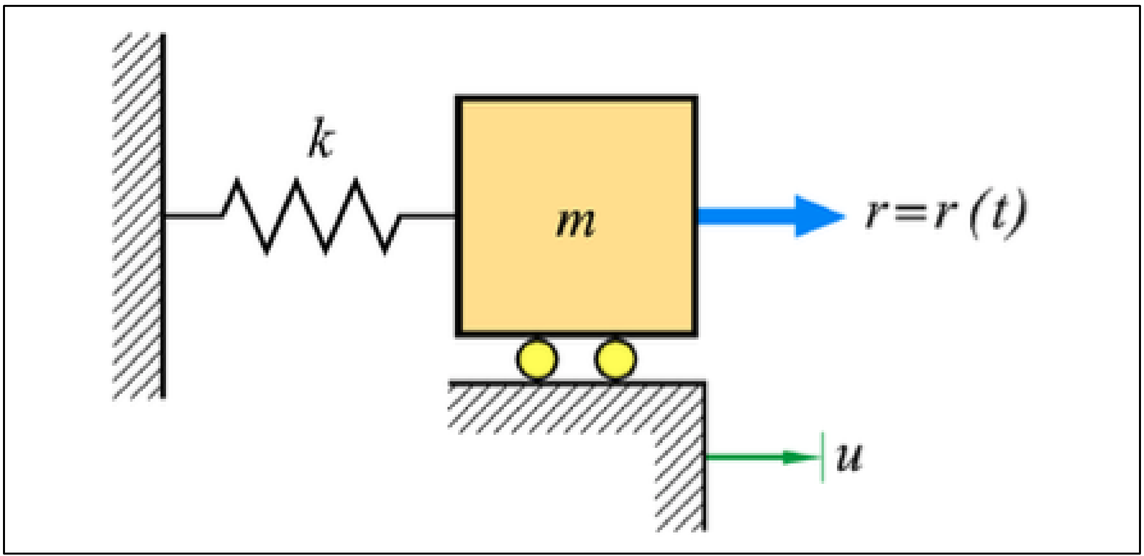
# SUMMARY

## Question 1

1. In the dynamic system of slide 6, let *r(t)* be a constant force *F*. What is the effect of *F* on the time-dependent displacement *u(t)* and the natural frequency of vibration of the system?




The differential equation which governs this harmonic system is

$$ku + m\ddot{u} = F$$

The solution of this Non-homogeneous ODE can be represented as the superposition of the homogeneous solution and the particular solution.

$$u(t) = u_h + u_p$$

For this particular ODE the homogeneous solution takes the form

$$u_h = A\,sin(\omega t) + B\,cos(\omega t)$$

And taking into account that F has no time dependence, the particular solution can be obtained through a constant

$$u_p = u_0\,;\ \dot{u}_p = 0;\ \ddot{u}_p = 0$$

Therefore,

$$ku_0 = F \rightarrow u_p = \frac{F}{k}$$

Superposing this result with the one of the homogeneous solution

$$u(t) = A\,sin(\omega t) + B\,cos(\omega t) + \frac{F}{k}$$

Applying the initial conditions $u(0) = 0$ and $\dot{u}(0) = 0$

$$A = 0\,, B = -\frac{F}{k}$$

Finally the expression of u(t) results as

$$u(t) = \frac{F}{k}(1 - \cos(\omega t))$$

Being $\omega = \sqrt{\frac{K}{m}}$ the natural frequency of the system

As can be seen, the constant force F does not affect the natural frequency of the system. Meanwhile the amplitude of the system is clearly dependent of the force magnitude.

## Question 2

> 2. A weight whose mass is $m$ is placed at the middle of a uniform axial bar of length $L$ that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of $m$, $L$, $E$ and $A$. Suggestion: First determine the effective $k$.

The effective stiffness of a beam can be analytically derived from the equation of the beam deflection when a punctual force is applied.

$$k = \frac{F}{\delta} = \frac{mg}{\delta}$$

In order to determine this parameter the deflection of the beam at the force application point is needed.

$$\delta\left(\frac{L}{2}\right) = \frac{FL^3}{192EI} = \frac{mgL^3}{192EI}$$

Substituting this expression in the one of the effective stiffness the estimation of the natural frequency results as

$$\omega = \sqrt{\frac{192EI}{mL^3}}$$

Being I the inertia of the cross section which depends on the area A.

## Question 3

> 3. Use the expression on slide 18 to derive the mass matrix of slide 17.

The general form of the consistent element mass matrix is

$$M = \int_{l^e} \rho N^T N dV$$

The shape functions are

$$N_1 = 1 - \xi\ ;\ N_2 = \xi$$

Therefore, integrating respect to the reference coordinates, the integral results as

$$M = \rho AL \int_0^1 \begin{bmatrix} (1-\xi)^2 & (1-\xi)\xi \\ (1-\xi)\xi & \xi^2 \end{bmatrix} d\xi$$

And finally, the result of this integral is the same as the indicated in the slide 19

$$M = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

## Question 4

> 4. **Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from $A_1$ to $A_2$.**

The formulation to compute the cross sectional area is the following

$$A = N_1 A_1 + N_2 A_2$$

Being the shape functions analogous to the previous linear ones employed in the question 3.

The expression of the mass matrix is again the general form

$$M = \int_{l^e} \rho N^T N dV$$

Now $dV$ depends on x and is expressed as $dV = A(x)dx$. If this differential is expressed in the reference coordinates the result of the integral is

$$M = \rho L \int_0^1 \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} [N_1 \quad N_2][N_1 A_1 + N_2 A_2] d\xi$$

Therefore,

$$M = \rho L \left( A_1 \int_0^1 \begin{bmatrix} (1-\xi)^3 & (1-\xi)^2\xi \\ (1-\xi)^2\xi & (1-\xi)\xi^2 \end{bmatrix} d\xi + A_2 \int_0^1 \begin{bmatrix} (1-\xi)^2\xi & (1-\xi)\xi^2 \\ (1-\xi)\xi^2 & \xi^3 \end{bmatrix} d\xi \right)$$

The resultant mass matrix is

$$M = \frac{\rho L}{12}\left( A_1 \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} + A_2 \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \right) = \frac{\rho L}{12} \begin{bmatrix} 3A_1 + A_2 & A_1 + A_2 \\ A_1 + A_2 & A_1 + 3A_2 \end{bmatrix}$$

If $A_1 = A_2$ the expression of the constant section mass matrix is recovered

## Question 5

> 5. **A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element?**

The diagonal mass matrix of this particular element in 3D is the following lumped mas matrix, which is very useful in terms of saving computational costs.

$$M = \frac{\rho A L}{2} \boldsymbol{I}_6 = \frac{\rho A L}{2} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$