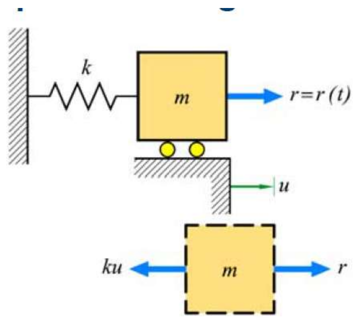
Assignment 10

In the dynamic system of slide 6, let r(t) be a constant force F. What is the effect of F on the time-dependent displacement u(t) and the natural frequency of vibration of the system?

The equation in the question is the following:



Newton's second law (f = ma) reads:

$$r - ku = m\ddot{u} \quad \text{or} \quad ku + m\ddot{u} = r$$

According to the solution of this differential linear equation will be the sum of a particular solution and a general solution. The general solution of the homogeneous equation is an harmonic movement and will have the form of an oscillatory function whose phase and amplitude are determined by initial conditions, that is,

$$u_h(t) = A\sin(wt + \varphi)$$

- w is the angular speed, proportional to the frequency.
- A is the amplitude, dependent on initial conditions.
- φ is the phase, also dependent on initial conditions.

Now we must solve the particular equation.

$$u_p(t) = \frac{F}{K}$$

Now the solution to the system is the superposition of uh and up. It therefore looks like:

$$u_h(t) = \frac{F}{K} + A\sin(wt + \varphi)$$

A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m, L, E and A. Suggestion: First determine the effective k.

The mass placed at the middle of the axial bar is the unique force on it, so the maximum displacement of a punctual load in the middle length of a clamped bar is:

$$\delta_{max}(t) = \frac{FL^3}{192EI}$$

where F is the force generated by the mass, L is the length of the bar, E is the Young's Modulus and I is the inertia of the bar in the direction of interest., assuming the bar is a square: $I = \frac{A^2}{12}$ and $F = mg$

$$\delta_{max} = \frac{mgL^3}{16EIA^2}$$

Then the stiffness k is defined as the force to apply to generate a unite movement.

$$k = \frac{F}{\delta_{max}} = \frac{16EA^2}{L^3}$$

So, the frequency of the structure is

$$w = \sqrt{\frac{16EA^2}{mL^3}}$$

Use the expression on slide 18 to derive the mass matrix of slide 17.

If the same shape functions used in the derivation of the stiffness matrix are chosen, the matrix is called the consistent mass matrix. For the 2-node prismatic bar element moving along x, the stiffness shape functions are

$$N_1(\xi) = \frac{1-\xi}{2}$$

$$N_2(\xi) = \frac{1+\xi}{2}$$

$$m = \int N^T N\rho dv$$

$$m = \rho A \int_{-1}^{1} \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} [1-\xi \quad \xi] d\xi = \frac{\rho AL}{6} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

which is equivalent to the one found in slide 19.

Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A1 to A2.

Now we are asked to derive the consistent mass matrix for a 2-node tapered bar element of length l and constant mass density ρ, moving along its axis x, if the cross section area varies as

$$A(\xi) = \frac{A_1(1-\xi)}{2} + \frac{A_2(1+\xi)}{2}$$

Now it is simply required to introduce this expression in the previous equation. We obtain the following:

$$m = \rho A \int_{-1}^{1} \frac{A_1(1-\xi)}{2} + \frac{A_2(1+\xi)}{2} \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} [1-\xi \quad \xi] d\xi$$

$$m = \frac{\rho L}{12} \begin{bmatrix} 3A_1 + A_2 & A_1 + A_2 \\ A_1 + A_2 & 3A_2 + A_1 \end{bmatrix}$$

A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element?

According to the slide 16 the mass matrix of a 3D two-noded bar is:

$$m = \frac{\rho AL}{2} I_6 = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$