Computational Structural Mechanics and Dynamics

As1 The Direct stiffness Method

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Assignment 1

Consider the truss problem defined in the Figure 1.1. All geometric and material properties: L, α, E *and* A as well as the applied forces P *and* H, are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed-displacement conditions at nodes $2, 3$ *and* 4. This structure is statically indeterminate as long as $\alpha \neq 0$.

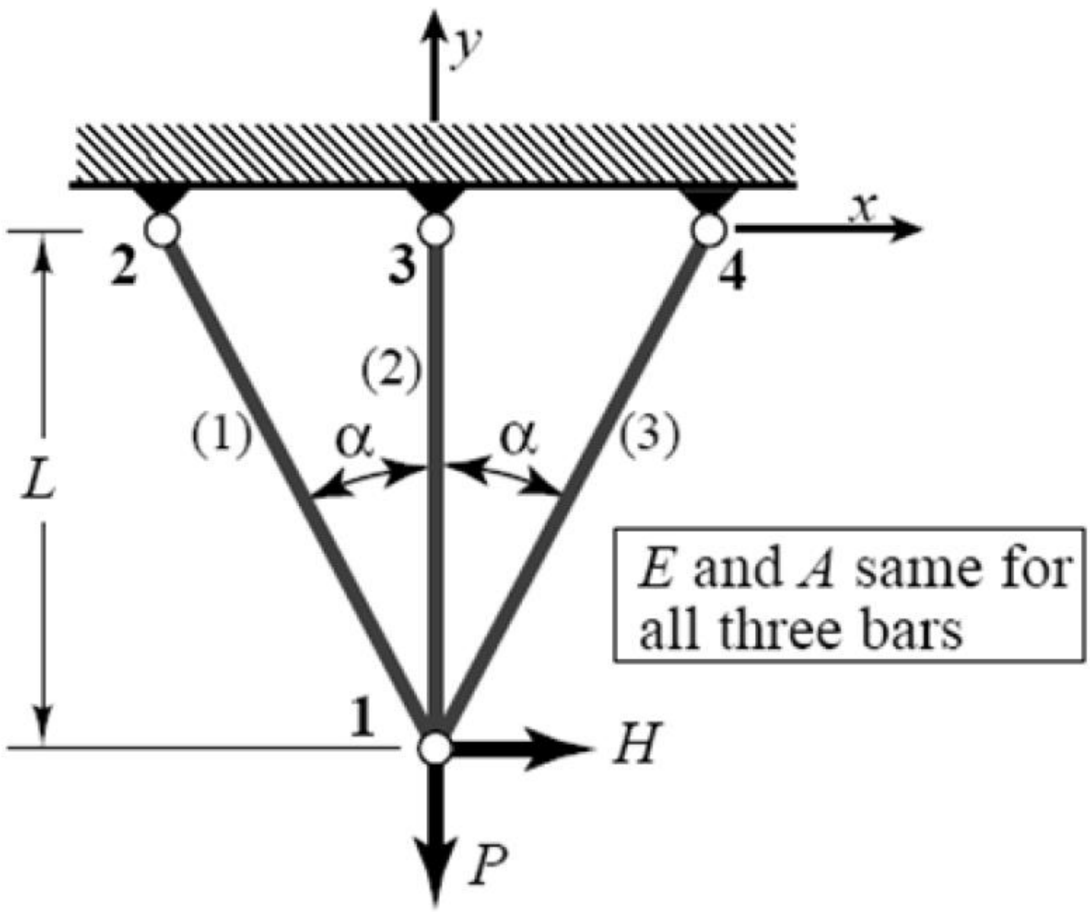


Figure 1.1 Truss structure, Geometry and mechanical features

(a) Show that the master stiffness equations are

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -cs^2 & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ Symm & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \\ u_{x4} \\ u_{x5} \\ u_{x6} \\ u_{x7} \\ u_{x8} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

In which $c = \cos(\alpha)$ *and* $s = \sin(\alpha)$. Explain from physics why the 5th row and column contain only zeros.

[Answer]

The elemental stiffness matrix in global form are

$$K^e = \frac{E^e A^e}{L^e}\begin{bmatrix} c_e^2 & s_e c_e & -c_e^2 & -s_e c_e \\ s_e c_e & s_e^2 & -s_e c_e & -s_e^2 \\ -c_e^2 & -s_e c_e & c_e^2 & s_e c_e \\ -s_e c_e & s_e c_e & s_e c_e & s_e^2 \end{bmatrix}$$

Where $s\ and\ c$ represent the sin and cos of the angle β of rotation of the local axis over the global axis.

From the problem statement and the choice of the local axis explained as $\beta_1 = \frac{\pi}{2} + \alpha, \beta_2 = \frac{\pi}{2}\ and\ \beta_3 = \frac{\pi}{2} - \alpha$

$$s_1 = sin\beta_1 = \sin\left(\frac{\pi}{2} + \alpha\right) = \cos(\alpha) = c$$

$$c_1 = cos\beta_1 = \cos\left(\frac{\pi}{2} + \alpha\right) = -\sin(\alpha) = -s$$

$$s_2 = sin\beta_2 = \sin\left(\frac{\pi}{2}\right) = 1$$

$$c_2 = cos\beta_2 = \cos\left(\frac{\pi}{2}\right) = 0$$

$$s_3 = sin\beta_3 = \sin\left(\frac{\pi}{2} - \alpha\right) = \cos(\alpha) = c$$

$$c_3 = cos\beta_3 = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin(\alpha) = s$$

$$K^1 = \frac{E^1 A^1}{L^1}\begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix} = \frac{EA}{L/c}\begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}$$

$$K^2 = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K^3 = \frac{EA}{L/c}\begin{bmatrix} s^2 & cs & -s^2 & -cs \\ cs & c^2 & -cs & -c^2 \\ -s^2 & -cs & s^2 & cs \\ -cs & -c^2 & -cs & c^2 \end{bmatrix}$$

The assembly of the global matrix is the following:

$$K = \frac{EA}{L}\begin{bmatrix} K_{11}^1 + K_{11}^2 + K_{11}^3 & K_{12}^1 + K_{12}^2 + K_{12}^3 & K_{13}^1 & K_{14}^1 & K_{13}^2 & K_{14}^2 & K_{13}^3 & K_{14}^3 \\ & K_{22}^1 + K_{22}^2 + K_{22}^3 & K_{23}^1 & K_{24}^1 & K_{23}^2 & K_{24}^2 & K_{23}^3 & K_{24}^3 \\ & & K_{33}^1 & K_{34}^1 & 0 & 0 & 0 & 0 \\ & & & K_{44}^1 & 0 & 0 & 0 & 0 \\ & & & & K_{33}^2 & K_{34}^2 & 0 & 0 \\ & & & & & K_{44}^2 & 0 & 0 \\ & & & & & & K_{33}^3 & K_{34}^3 \\ Symm & & & & & & & K_{44}^3 \end{bmatrix}$$

$$
= \frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^3 & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ Symm & & & & & & & c^3 \end{bmatrix}
$$

From the boundary condition, we have $fx_1 = H, fy_1 = -P$

$$
f = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Then, we obtain the equation (1).

The 5th row and column contains only zeros because the interforce of member (2) only have fore in y direction while there is no force in x direction. So, $fx_3 \equiv 0$. To assure it, all the 5th row and column must contain only zeros.

(b) Apply the BC's and show the 2-equation modified stiffness system

[Answer]

Nodes 2, 3 and 4 are restrained with a fixed displacement of 0. The only non-restrained degrees of freedom are $u_{x1}\ and\ u_{x2}$. It is translated in the master equations eliminating all equations but first and second one. Meanwhile, eliminating all the unknowns expect $u_{x1}\ and\ u_{x2}$.As the prescribed displacements are 0, there is no contribution on the force vector due to the displacement boundary conditions.

The resulting system is:

$$
\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}
$$

(c) Solve for the displacement $u_{x1}\ and\ u_{x2}$. Check that the solution makes physical sense for the limit cases $\alpha \to 0\ and\ \alpha \to \pi/2$. Why dose u_{x1} 'blow up' if $H \neq 0$ and $\alpha \to 0$?

[Answer]

The result system is diagonal, so the solution is straight:

$$
u_{x1} = \frac{L}{EA}\frac{H}{2cs^2}
$$

$$
u_{y1} = \frac{L}{EA}\frac{-P}{1+2c^3}
$$

In the limit case when $\alpha \to 0$,

$$
c \to 1\ and\ s \to 0, then\ u_{x1} \to \infty\ and\ u_{y1} \to \frac{-P}{3}\frac{L}{EA}
$$

In this case, the force H generate a moment around the coordinate origin node (node 3). The reactions to that moment are provided by member 1 and 3. While $\alpha \to 0$, the nodes2,3 and 4 are close to each other. This is reducing the lever arm. For this reason, the axial force $f^{(1)}\ and\ f^{(3)}$ become bigger to provide the same reaction moment which cause a larger deformation. So, while $\alpha \to 0,\ u_{x1} \to \infty$.

In the limit case when $\alpha \to \pi/2$,

$$c \to 0\ and\ s \to 1, then\ u_{x1} \to \infty\ and\ u_{y1} \to \frac{PL}{EA}$$

In this case, the solution also tends to infinity for the same reason. The member 1 and 3 cannot compensate the moment produced by H. This is due to the fact that while the length of the element tends to infinity, its stiffness is reduced to 0.

(d) Recover the axial forces in the three members. Partial answer: $F^{(3)} = \frac{-H}{2s} + P\frac{c^2}{1+2c^3}$. Why do $F^{(1)}\ and\ F^{(3)}$ 'blow up' if $H \neq 0$ and $\alpha \to 0$?

[Answer]

$$\bar{f}^e = \bar{K}^e \bar{u}^e = \bar{K}^e T^e u^e$$

Element 1:

$$\frac{EA}{L/c}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & c_1 & s_1 \\ 0 & 0 & s_1 & c_1 \end{bmatrix}\begin{bmatrix} u_{x1}^1 \\ u_{y1}^1 \\ u_{x2}^1 \\ u_{y2}^1 \end{bmatrix}$$

$$= \frac{EA}{L/c}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -s & c & 0 & 0 \\ c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & c & -s \end{bmatrix}\begin{bmatrix} \frac{L}{EA}\frac{H}{2cs^2} \\ \frac{L}{EA}\frac{-P}{1+2c^3} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-H}{2s} - P\frac{c^2}{1+2c^3} \\ 0 \\ \frac{H}{2s} + P\frac{c^2}{1+2c^3} \\ 0 \end{bmatrix}$$

$$F^{(1)} = \frac{H}{2s} + P\frac{c^2}{1+2c^3}$$

Element 2:

$$\frac{EA}{L/c}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_2 & s_2 & 0 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & c_2 & s_2 \\ 0 & 0 & s_2 & c_2 \end{bmatrix}\begin{bmatrix} u_{x1}^2 \\ u_{y1}^2 \\ u_{x2}^2 \\ u_{y2}^2 \end{bmatrix}$$

$$= \frac{EA}{L/c}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{L}{EA}\frac{H}{2cs^2} \\ \frac{L}{EA}\frac{-P}{1+2c^3} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-P}{1+2c^3} \\ 0 \\ \frac{P}{1+2c^3} \\ 0 \end{bmatrix}$$

$$F^{(2)} = \frac{P}{1+2c^3}$$

Element 3:

$$\frac{EA}{L/c}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_3 & s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & c_3 & s_3 \\ 0 & 0 & s_3 & c_3 \end{bmatrix}\begin{bmatrix} u_{x1}^3 \\ u_{y1}^3 \\ u_{x2}^3 \\ u_{y2}^3 \end{bmatrix}$$

$$= \frac{EA}{L/c}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} s & c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & c & s \end{bmatrix}\begin{bmatrix} \frac{L}{EA}\frac{H}{2cs^2} \\ \frac{L}{EA}\frac{-P}{1+2c^3} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{H}{2s} - P\frac{c^2}{1+2c^3} \\ 0 \\ -\frac{H}{2s} + P\frac{c^2}{1+2c^3} \\ 0 \end{bmatrix}$$

$$F^{(3)} = \frac{-H}{2s} + P\frac{c^2}{1+2c^3}$$

While $H \neq 0$ and $\alpha \to 0,\ c \to 1\ and\ s \to 0$. Then $F^{(1)} \to \infty\ and\ F^{(3)} \to -\infty$. The reason is the same as the one in the question(c).

Assignment 2

Dr. who propose "improving" the result for the example truss of the 1st lesson by putting one extra node 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members:(3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr. who's suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.

[Answer]

The elemental matrix of elements (1) and (2) used in this problem are the same than in the example. The matrix (3) and (4) are just the double of (3) in the example as the elements are equal but half length. So, the global stiffness matrix is:

$$K = \begin{bmatrix} K_{11}^1 + K_{11}^3 & K_{12}^1 + K_{12}^3 & K_{13}^1 & K_{14}^1 & 0 & 0 & K_{13}^3 & K_{14}^3 \\ & K_{22}^1 + K_{22}^3 & K_{23}^1 & K_{24}^1 & 0 & 0 & K_{23}^3 & K_{24}^3 \\ & & K_{33}^1 + K_{11}^2 & K_{34}^1 + K_{12}^2 & K_{13}^2 & K_{14}^2 & 0 & 0 \\ & & & K_{44}^1 + K_{22}^2 & K_{23}^2 & K_{24}^1 & 0 & 0 \\ & & & & K_{33}^2 + K_{11}^4 & K_{34}^2 + K_{12}^4 & K_{23}^4 & K_{14}^4 \\ & & & & & K_{44}^2 + K_{22}^4 & K_{23}^4 & K_{24}^4 \\ & & & & & & K_{33}^3 + K_{33}^4 & K_{34}^3 + K_{34}^4 \\ Symm & & & & & & & K_{33}^3 + K_{33}^4 \end{bmatrix}$$

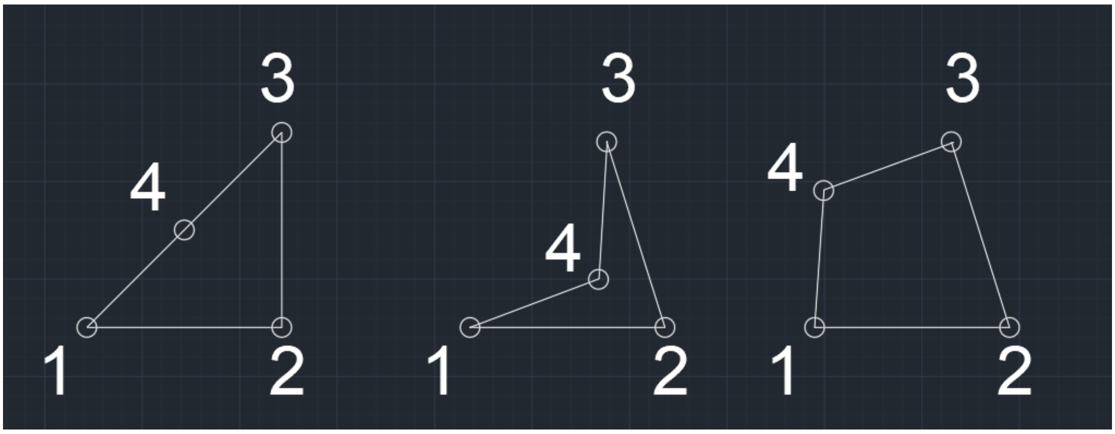
Consider the BC, $u_{x1} = u_{y1} = u_{y2} = 0$

We obtain

$$\hat{K} = \begin{bmatrix} K_{33}^1 + K_{11}^2 & K_{13}^2 & K_{14}^2 & 0 & 0 \\ & K_{33}^2 + K_{11}^4 & K_{34}^2 + K_{12}^4 & K_{23}^4 & K_{14}^4 \\ & & K_{44}^2 + K_{22}^4 & K_{23}^4 & K_{24}^4 \\ & & & K_{33}^3 + K_{33}^4 & K_{34}^3 + K_{34}^4 \\ Symm & & & & K_{33}^3 + K_{33}^4 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 5 & 5 & -5 & -5 \\ 0 & 5 & 10 & -5 & -5 \\ 0 & -5 & -5 & 10 & 10 \\ 0 & -5 & -5 & 10 & 10 \end{bmatrix}$$

Obviously, the matrix is singular. That is mean the problem is not well posed. Because the system is under-constrained. The bar element can only be used in full triangulated trusses as they are not able to resist shear forces nor bending moments.

The physical reason for the matrix is singular is that the structure is unstable. The following figure is the example that this structure is unstable.



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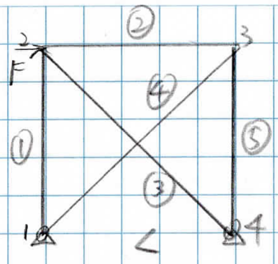

$$K^e = \left(\frac{EA}{L}\right)^e \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ & s^2 & -sc & -s^2 \\ & & c^2 & sc \\ & & & s^2 \end{bmatrix}$$

$L = 6\ m \quad A = 6\ cm^2$

$E = 200\ GPa \quad F = 80\ KN$

$s=1 \quad c=0$

$$K_1^e = \left(\frac{EA}{L_1}\right)^1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ & & & 1 \end{bmatrix}$$

$s=0 \quad c=1$

$$K_2^e = \left[\frac{EA}{L_2}\right]^2 \begin{bmatrix} 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ & & & 0 \end{bmatrix}$$

$s=-\frac{\sqrt{2}}{2} \quad c=\frac{\sqrt{2}}{2}$

$$K_3 = \left[\frac{EA}{L_3}\right]^3 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ & & \frac{1}{2} & -\frac{1}{2} \\ & & & \frac{1}{2} \end{bmatrix}$$

$s=\frac{\sqrt{2}}{2} \quad c=\frac{\sqrt{2}}{2}$

$$K_4 = \left[\frac{EA}{L_4}\right]^4 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ & & \frac{1}{2} & \frac{1}{2} \\ & & & \frac{1}{2} \end{bmatrix}$$

$s=-1 \quad c=0$

$$K_5 = \left[\frac{EA}{L_5}\right]^5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ & & & 1 \end{bmatrix}$$

Elemen 1

$$\begin{bmatrix} f_{x_1}^{(1)} \\ f_{y_1}^{(1)} \\ f_{x_2}^{(1)} \\ f_{y_2}^{(1)} \\ f_{x_3}^{(1)} \\ f_{y_3}^{(1)} \\ f_{x_4}^{(1)} \\ f_{y_4}^{(1)} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}$$

Element 2

$$\begin{bmatrix} f^{(2)}_{x_1} \\ f^{(2)}_{y_1} \\ f^{(2)}_{x_2} \\ f^{(2)}_{y_2} \\ f^{(2)}_{x_3} \\ f^{(2)}_{y_3} \\ f^{(2)}_{x_4} \\ f^{(2)}_{y_4} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}$$

Element 3

$$\begin{bmatrix} f^{(3)}_{x_1} \\ f^{(3)}_{y_1} \\ f^{(3)}_{x_2} \\ f^{(3)}_{y_2} \\ f^{(3)}_{x_3} \\ f^{(3)}_{y_3} \\ f^{(3)}_{x_4} \\ f^{(3)}_{y_4} \end{bmatrix} = \frac{EA}{\sqrt{2}L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}$$

Element 4

$$\begin{bmatrix} f^{(4)}_{x_1} \\ f^{(4)}_{y_1} \\ f^{(4)}_{x_2} \\ f^{(4)}_{y_2} \\ f^{(4)}_{x_3} \\ f^{(4)}_{y_3} \\ f^{(4)}_{x_4} \\ f^{(4)}_{y_4} \end{bmatrix} = \frac{EA}{\sqrt{2}L} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}$$

Element 5

$$\begin{bmatrix} f^{(5)}_{x_1} \\ f^{(5)}_{y_1} \\ f^{(5)}_{x_2} \\ f^{(5)}_{y_2} \\ f^{(5)}_{x_3} \\ f^{(5)}_{y_3} \\ f^{(5)}_{x_4} \\ f^{(5)}_{y_4} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}$$

Assembly

$$f = f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)} + f^{(5)}$$

$$\begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} = 80 \\ f_{y_2} = 0 \\ f_{x_3} = 0 \\ f_{y_3} = 0 \\ f_{x_4} \\ f_{y_4} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 0.3536 & 0.3536 & 0 & 0 & -0.3536 & -0.3536 & 0 & 0 \\ 0.3536 & 1.3536 & 0 & -1 & -0.3536 & -0.3536 & 0 & 0 \\ 0 & 0 & 1.3536 & -0.3536 & -1 & 0 & -0.3536 & 0.3536 \\ 0 & -1 & -0.3536 & 1.3536 & 0 & 0 & 0.3536 & -0.3536 \\ -0.3536 & -0.3536 & -1 & 0 & 1.3536 & 0.3536 & 0 & 0 \\ -0.3536 & -0.3536 & 0 & 0 & 0.3536 & 1.3536 & 0 & -1 \\ 0 & 0 & -0.3536 & 0.3536 & 0 & 0 & 0.3536 & -0.3536 \\ 0 & 0 & 0.3536 & -0.3536 & 0 & -1 & -0.3536 & 0.3536 \end{bmatrix} \cdot \begin{bmatrix} u_{x_1} = 0 \\ u_{y_1} = 0 \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} = 0 \\ u_{y_4} = 0 \end{bmatrix}$$

B.C. $u_{x_1} = u_{y_1} = u_{x_4} = u_{y_4} = 0 \quad f_{x_2} = 80 \quad f_{y_2} = f_{x_3} = f_{y_3} = 0$

$$\begin{bmatrix} 80 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.3536 & -0.3536 & -1 & 0 \\ -0.3536 & 1.3536 & 0 & 0 \\ -1 & 0 & 1.3536 & 0.3536 \\ 0 & 0 & 0.3536 & 1.3536 \end{bmatrix} \cdot \frac{200\times10^6 \cdot 6\times10^{-4}}{6} \begin{bmatrix} u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{bmatrix}$$

$$\begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 8.54139 \times 10^{-3} \\ 2.23103 \times 10^{-3} \\ 6.67242 \times 10^{-3} \\ -1.76897 \times 10^{-3} \end{bmatrix} m$$