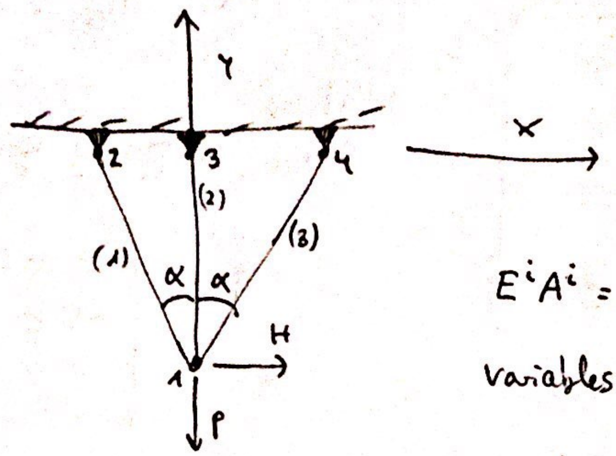
Student: Oscar Salvador

Computational Structural Mech. and Dynam.

Assignment 1

Direct Stiffness Method

(1)



$E^i A^i$ = constant

variables: L, α, E, A, P, H

$$L^{(2)} = L$$

$$L^{(1)} = L^{(3)} = \frac{L}{\cos\alpha}$$

8 DOF's

6 fixed → nodes 2, 3, 4 : $[u_{2x}, u_{2y}, u_{3x}, u_{3y}, u_{4x}, u_{4y}] = 0$

2 free → node 1 : u_{1x}, u_{1y}

2 prescribed forces: $f_{1y} = -P$, $f_{1x} = H$

Element stiffness equations $\quad \bar{f} = \bar{\bar{k}} \cdot \bar{u}$

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \left(\frac{EA}{L}\right)^e \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix}$$

Globalization

Transformation local $\rightarrow$ global coordinates

Displacements

$\bar{u}_{xi}, \bar{u}_{yi} \rightarrow$ local

$u_{xi}, u_{yi} \rightarrow$ global

$$\left.\begin{aligned}
\bar{u}_{xi} &= u_{xi}\,c + u_{yi}\,s \\
\bar{u}_{yi} &= -u_{xi}\,s + u_{yi}\,c \\
\bar{u}_{xj} &= u_{xj}\,c + u_{yj}\,s \\
\bar{u}_{yj} &= -u_{xj}\,s + u_{yj}\,c
\end{aligned}\right]$$

Matrix form $\bar{u}^e = T^e u^e$

$$T^e = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

Forces

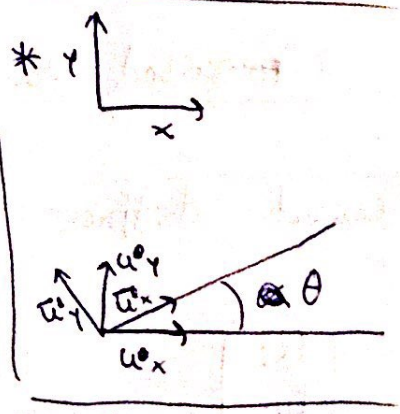
$$f^e = (T^e)^T \bar{f}^e \qquad (T_e)^T = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix}$$

As $\bar{k}^e \bar{u}^e = \bar{f}^e$ (local)

Plugging in the matrices stated above

$$k^e = (T^e)^T \, \bar{k}^e \, T^e$$

$$k^e = \left(\frac{EA}{L}\right)^e \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$



This is the transformation

therefore,

Elements:

① $\theta = 90 + \alpha$

② $\theta = 90$

③ $\theta = 90 - \alpha$

$\rightarrow$

$\cos(90+\alpha) = -\sin\alpha$

$\sin(90+\alpha) = \cos\alpha$

$\cos(90-\alpha) = \sin\alpha$

$\sin(90-\alpha) = \cos\alpha$

$\cos(90) = 0$

$\sin(90) = 1$

Therefore, the global (element) stiffness matrices are

$$K^{(1)} = \frac{EA}{L/c}\begin{bmatrix} s^2 & -sc & -s^2 & sc & 0 & 0 & 0 & 0 \\ -sc & c^2 & sc & -c^2 & 0 & 0 & 0 & 0 \\ -s^2 & sc & s^2 & -sc & 0 & 0 & 0 & 0 \\ sc & -c^2 & -sc & c^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^{(2)} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^{(3)} = \frac{EA}{L/c}\begin{bmatrix} s^2 & sc & 0 & 0 & 0 & 0 & -s^2 & -sc \\ sc & c^2 & 0 & 0 & 0 & 0 & -sc & -c^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -s^2 & -sc & 0 & 0 & 0 & 0 & s^2 & sc \\ -sc & -c^2 & 0 & 0 & 0 & 0 & sc & c^2 \end{bmatrix}$$

Assembly matrix

$$K = \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2 s & 0 & 0 & -cs^2 & -c^2 s \\ & 1+2c^3 & c^2 s & -c^3 & 0 & -1 & -c^2 s & -c^3 \\ & & cs^2 & -c^2 s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & \text{symm} & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2 s \\ & & & & & & & c^3 \end{bmatrix}$$

master stiffness equations

$$K u = f \qquad u = \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \\ u_{4x} \\ u_{4y} \end{bmatrix} \qquad f = \begin{array}{c} \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} f_{1x} \\ f_{1y} \\ \\ \\ \\ \\ \\ \\ \end{array}$$

2 Boundary conditions

Prescribed displacements: $[u_{2x}, u_{2y}, u_{3x}, u_{3y}, u_{4x}, u_{4y}] = 0$

Prescribed forces : $f_{1x} = H \qquad f_{1y} = -P$

Applying these BCs the system results as

$$\frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & \text{symm} & & & & 1 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{1y} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

& Gauss elimination

&

$$\frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{1y} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

③ Solution of the 2×2 system

$$\frac{EA}{L} 2cs^2 \cdot u_{1x} = H \rightarrow \boxed{u_{1x} = \frac{L}{EA}\frac{H}{2cs^2}}$$

$$(1+2c^3)\frac{EA}{L} \cdot u_{1y} = -P \rightarrow \boxed{u_{1y} = -\frac{L}{EA}\frac{P}{1+2c^3}}$$

when $\alpha \rightarrow 0$:

$$\lim_{\alpha\to 0} u_{1x} = +\infty \qquad \lim_{\alpha\to 0} u_{1y} = -\frac{LP}{3EA}$$

In ~~this~~ this case, the three bars ~~act~~ tend to be in a vertical position and the structure can not support horitzontal forces ($H \neq 0$).

If $H=0 \rightarrow \lim_{\alpha\to 0} u_{1x} = \frac{0}{0}$ (Indetermined)

when $\alpha \rightarrow \pi/2$:

$$\lim_{\alpha\to\pi/2}(u_{1x}) = +\infty \qquad \lim_{\alpha\to\pi/2} u_{1y} = -\frac{LP}{EA}$$

The physical explanation is that the system is equivalent to a single vertical bar. The reason of such behaviour is that the bars 1 and 3 tend to be horitzontal and with infinite length. The results is the same pendulum behaviour ~~than~~ of ~~a~~ the previous case.

④ $\boxed{\text{Axial forces}}$

In order to calculate the axial forces it is needed to start withe the consitutive equation of 1D elastic problems.

$$\sigma = E\varepsilon \quad \text{(constitutive eq.)}$$

$$F/A = \frac{Eu}{L} \rightarrow F = \frac{EA}{L} u$$

This u is the axial displacement

$\boxed{\text{Element 1}}$

$$F^{(1)} = \frac{EA}{L/c}\left(-u_{1x}\, s + u_{y1}\, c\right) = \frac{H}{2s} + \frac{c^2 P}{1+2c^3}$$

$\boxed{\text{Element 2}}$

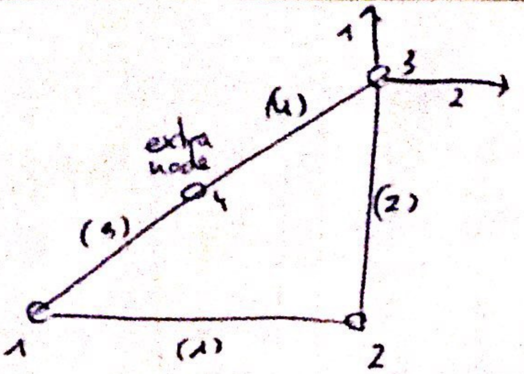
$$F^{(2)} = \frac{EA}{L} u_{y1} = -\frac{P}{1+2c^3}$$

$\boxed{\text{Element 3}}$

$$F^{(3)} = \frac{EA}{L/c}\left(u_{1x}\, s + u_{y1}\, c\right) = -\frac{H}{2s} + \frac{c^2 P}{1+2c^3}$$

If $H \neq 0$ and $\alpha \rightarrow 0$ there will be no stiffness in the horitzontal axis (pendulum)

5 putting and extra node



Element stiffness equations

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix} = \begin{bmatrix} 10 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 10 \end{bmatrix} \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix}$$

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x4}^{(3)} \\ f_{y4}^{(3)} \end{bmatrix} = \begin{bmatrix} 20 & 20 & -20 & -20 \\ 20 & 20 & -20 & -20 \\ -20 & -20 & 20 & 20 \\ -20 & -20 & 20 & 20 \end{bmatrix} \begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x4}^{(3)} \\ u_{y4}^{(3)} \end{bmatrix}$$

$$\begin{bmatrix} f_{x3}^{(4)} \\ f_{y3}^{(4)} \\ f_{x4}^{(4)} \\ f_{y4}^{(4)} \end{bmatrix} = \begin{bmatrix} 20 & 20 & -20 & -20 \\ 20 & 20 & -20 & -20 \\ -20 & -20 & 20 & 20 \\ -20 & -20 & 20 & 20 \end{bmatrix} \begin{bmatrix} u_{x3}^{(4)} \\ u_{y3}^{(4)} \\ u_{x4}^{(4)} \\ u_{y4}^{(4)} \end{bmatrix}$$

Assembly

$$
\begin{bmatrix}
30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\
20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\
-10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 10 & 0 & -10 & 0 & 0 \\
0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\
0 & 0 & 0 & -10 & 20 & 30 & -20 & -20 \\
-20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \\
-20 & -20 & 0 & 0 & -20 & -20 & 40 & 40
\end{bmatrix}
\begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \\ u_{4x} \\ u_{4y} \end{bmatrix}
=
\begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{bmatrix}
$$

BCs

$f_{3x} = 2$ $\qquad [u_{1x}, u_{1y}, u_{2y}] = 0$

$f_{3y} = 1$

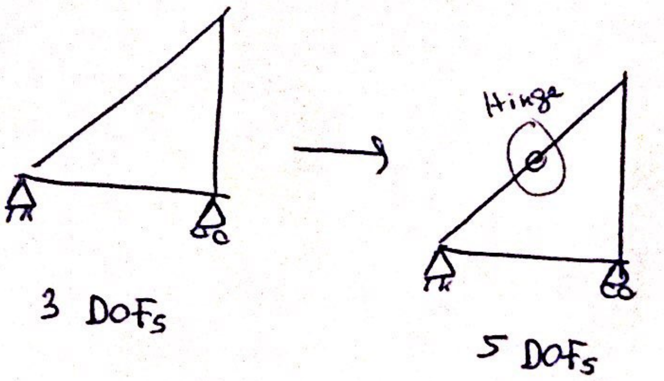
$\downarrow$ Reducing

$$
\begin{bmatrix}
10 & 0 & 0 & 0 & 0 \\
0 & 20 & 20 & -20 & -20 \\
0 & 20 & 30 & -20 & -20 \\
0 & -20 & -20 & 40 & 40 \\
0 & -20 & -20 & 40 & 40
\end{bmatrix}
\begin{bmatrix} u_{2x} \\ u_{3x} \\ u_{3y} \\ u_{4x} \\ u_{4y} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}
$$

Identical rows $\rightarrow$ $\boxed{\text{singular matrix}}$

Physical explanation

Adding a new node the number of DOFs has increased from 3 to 5.



Considering these BCs, only three DOFs are fixed, and thus the ~~structure~~ previous structure now is a mechanism.

With this example, it is shown that by increasing the nodes we do not achieve better results, but we introduce new DOFs that ~~do not~~ modify the behaviour of the structure.