Assignment 1.1

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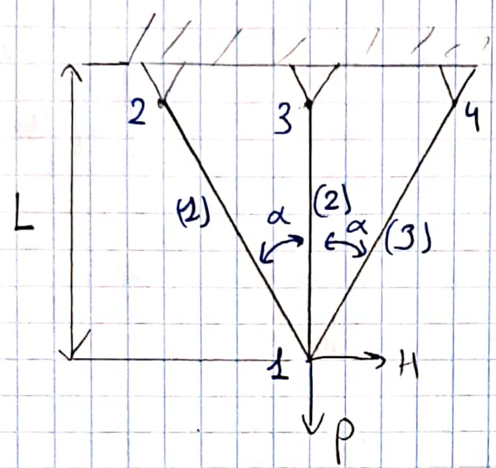



E, A same for all three bars

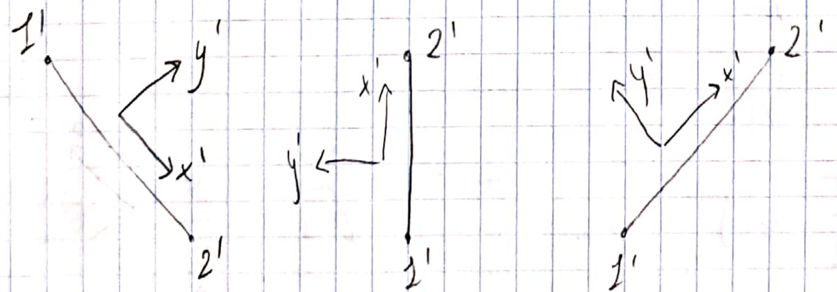
1.

$c = \cos\alpha$

$s = \sin\alpha$

Master stiffness equation:

First step is to evaluate each bar locally with local coordinates and transform them to the global coordinates.



* Considering local x-axis alligned with the bar and Hooke's law, it is concluded:

$$\underline{K}^{(e)} \cdot \underline{u}^{(e)} = \frac{E^{(e)} \cdot A^{(e)}}{L^{(e)}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \bar{u}_{1x} \\ \bar{u}_{1y} \\ \bar{u}_{2x} \\ \bar{u}_{2y} \end{bmatrix} = \underline{f}$$

* Taking into account $\phi$ is the inclination of the bar in relation to global x-coordinate.

$$\underline{u}^{e} = T^{e} u^{e} = \begin{bmatrix} \cos\phi^{e} & \sin\phi^{e} & 0 & 0 \\ -\sin\phi^{e} & \cos\phi^{e} & 0 & 0 \\ 0 & 0 & \cos\phi^{e} & \sin\phi^{e} \\ 0 & 0 & -\sin\phi^{e} & \cos\phi^{e} \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix}$$

$i, j \rightarrow$ global numbering of nodes

* Same for node forces

$$f^{e} = (T^{e})^{T} \underline{f}^{e} = K^{(e)} u^{(e)} = (T^{e})^{T} \cdot \underline{K}^{e} \cdot T^{e}$$

* For this problem:

$$\phi_1 = -\left(\frac{\pi}{2} - \alpha\right), \quad \phi_2 = \frac{\pi}{2}, \quad \phi_3 = \frac{\pi}{2} - \alpha$$

and using,

$$c = \cos(\alpha), \quad s = \sin(\alpha)$$

# Element 1:

$$\sin(\phi_1) = \sin\left(\frac{\pi}{2} + \alpha\right) = \cos(\alpha) = c$$

$$\cos(\phi_1) = \cos\left(\frac{\pi}{2} + \alpha\right) = -\sin(\alpha) = -s$$

$$K^{(1)} = \frac{EA}{L}\begin{bmatrix} cs^2 & -sc^2 & -cs^2 & sc^2 \\ & c^3 & sc^2 & -c^3 \\ & & cs^2 & -sc^2 \\ \text{SYM} & & & c^3 \end{bmatrix}; \quad \text{length of the bar} = \frac{L}{c}$$

# Element 2

$$\sin(\phi_2) = 1 \quad ; \quad \cos(\phi_2) = 0$$

$$K^{(2)} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

# Element 3:

$$\sin(\phi_3) = \sin\left(\frac{\pi}{2} - \alpha\right) = \cos(\alpha) = c$$

$$\cos(\phi_3) = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin(\alpha) = s$$

$$K^{(3)} = \frac{EA}{L}\begin{bmatrix} cs^2 & sc^2 & -cs^2 & -sc^2 \\ & c^3 & -sc^2 & -c^3 \\ & & cs^2 & sc^2 \\ \text{SYM} & & & c^3 \end{bmatrix}; \quad \text{length of the bar} = \frac{L}{c}$$

+ Global matrix:

$$
K = \frac{EA}{L}\begin{bmatrix}
K_{11}^{(1)}+K_{11}^{(2)}+K_{11}^{(3)} & K_{12}^{(1)}+K_{12}^{(2)}+K_{12}^{(3)} & K_{13}^{(1)} & K_{14}^{(1)} & K_{13}^{(2)} & K_{14}^{(2)} & K_{13}^{(3)} & K_{14}^{(3)} \\
 & K_{22}^{(1)}+K_{22}^{(2)}+K_{22}^{(3)} & K_{23}^{(1)} & K_{24}^{(1)} & K_{23}^{(2)} & K_{24}^{(2)} & K_{23}^{(3)} & K_{24}^{(3)} \\
 & & K_{33}^{(1)} & K_{34}^{(1)} & 0 & 0 & 0 & 0 \\
 & & & K_{44}^{(1)} & 0 & 0 & 0 & 0 \\
 & & & & K_{33}^{(2)} & K_{34}^{(2)} & 0 & 0 \\
 & & & & & K_{44}^{(2)} & 0 & 0 \\
 & & & & & & K_{33}^{(3)} & K_{34}^{(3)} \\
 & & & & & & & K_{44}^{(3)}
\end{bmatrix}
$$

+ Finally:

5th column

5th row

$$
K \cdot u = \frac{EA}{L}\begin{bmatrix}
2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\
 & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\
 & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\
 & & & c^3 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 \\
 & & & & & 1 & 0 & 0 \\
 & \text{sym} & & & & & cs^2 & c^2s \\
 & & & & & & & c^3
\end{bmatrix}
\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}
$$

+ Force vector $f$ is

$$
F = \begin{bmatrix} H & -P & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T
$$

since there are only external forces on node 1

# The fifth row and column are zero. The structural elements are bars, so they only can handle axial stresses.
As the second element is vertical, the horizontal reaction of the node 3 is 0.
The fifth row and column represent the x component of the reaction force on the node 3, so they must contain only 0.

2. Apply BC's

The truss system is fixed on the nodes 2, 3 and 4, so:

$$u_{x2} = u_{y2} = u_{x3} = u_{y3} = u_{x4} = u_{y4} = 0$$

Eliminating the rows and columns corresponding to these displacements:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

3. $u_{x1}, u_{y1}$ ?

Physical sense $\alpha \to 0$ , $\alpha \to \pi/2$

# From the previous system:

$$\left[ u_{x1} = \frac{HL}{2EAcs^2} \right]$$

$$\left[ u_{y1} = -\frac{PL}{EA(1+2c^3)} \right]$$

# There is physical sense for displacement on y-direction. For $\alpha \to 0$ all bars are aligned offering the highest resistance for deformation. $U_{y1}$ reaches its lowest value. Increasing $\alpha$, the displacement increases as well until its highest value for $\alpha \to \frac{\pi}{2}$.

# For displacement on x-direction, there is physical sense for intermidiate values of $\alpha$, but problems for the limit cases. There is no sense for $\alpha \to \frac{\pi}{2}$, because it would mean that the bars 1 and 3 were horizontal and infinite in length.

For $\alpha \to 0$, the bars would be alligned and jointed only one one point on the ceiling. After imposing $H \neq 0$, there would be no resistence for rotation and there could be no equilebrium. The bars would tend to rotate, so, the solution "blows up"!

4. Axial forces $F^{(3)} = -\frac{H}{2s} + \frac{P_c^2}{1+2c^3}$

$F^{(1)}$ and $F^{(3)}$ blows up if $H \neq 0$ and $\alpha \to 0$ ?

\+ Axial force:

$$F^{(e)} = \frac{E^{(e)} \cdot A^{(e)}}{L^{(e)}} \cdot d^{(e)} \quad ; \quad d^{(e)} = \underline{u}_{jx}^{(e)} - \underline{u}_{ix}^{(e)}$$

\+ To use local coordinates, $\underline{u}^{(e)} = T^e u^e$

$$\underline{u}^{(1)} = \begin{bmatrix} \underline{u}_{x_1}^{(1)} \\ u_{y_1}^{(1)} \\ \underline{u}_{x_2}^{(1)} \\ u_{y_2}^{(1)} \end{bmatrix} = \begin{bmatrix} s & -c & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{bmatrix} \cdot \begin{bmatrix} u_{x2}=0 \\ u_{y2}=0 \\ u_{x1} \\ u_{y1} \end{bmatrix}$$

$$\underline{u}^{(2)} = \begin{bmatrix} \underline{u}_{x_1}^{(2)} \\ \underline{u}_{y_1}^{(2)} \\ u_{x2}^{(2)} \\ u_{y_2}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x3}=0 \\ u_{y3}=0 \end{bmatrix}$$

$$\underline{u}^{(3)} = \begin{bmatrix} \underline{u}_{x_1}^{(3)} \\ u_{y_1}^{(3)} \\ \underline{u}_{x_2}^{(3)} \\ \underline{u}_{y_2}^{(3)} \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x4}=0 \\ u_{y4}=0 \end{bmatrix}$$

# Elongations:

$$d^{(1)} = (s\,u_{x1} - c\,u_{y1}) - 0 = \frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}$$

$$d^{(2)} = 0 - u_{y1} = \frac{PL}{EA(1+2c^3)}$$

$$d^{(3)} = 0 - (s\,u_{x1} + c\,u_{y1}) = -\frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}$$

So,

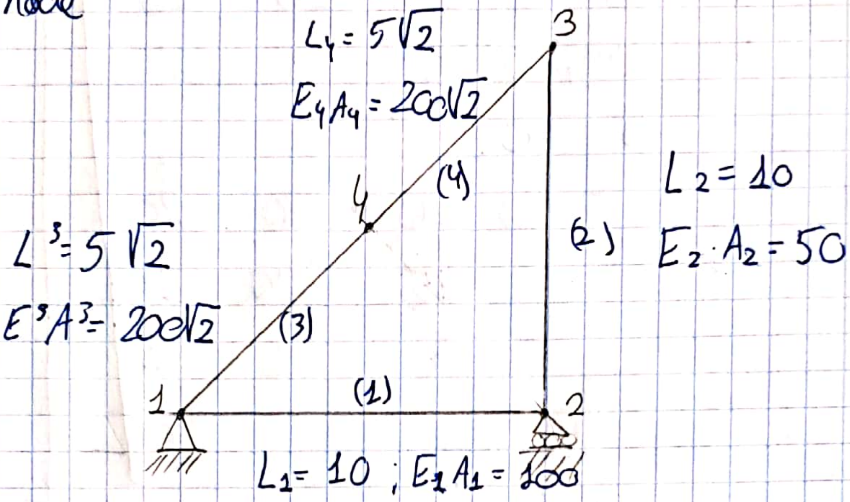
$$\left[ F^{(1)} = \frac{H}{2s} + \frac{Pc^2}{1+2c^3} \right] \qquad \left[ F^{(2)} = \frac{P}{(1+2c^3)} \right]$$

$$\left[ F^{(3)} = -\frac{H}{2s} + \frac{Pc^2}{1+2c^3} \right]$$

\+ For $\alpha \to 0$, the solution "blow up" for the bars 1 and 3. As explained before, this can be explained physically by the fact the system cannot hold equilibrium for this circumstance.
As the axial force would be perpendicular to the force $H \neq 0$, they cannot compensate the force in any bar. So, the solution tends to infinite and the bars would tend to rotate.

Assignment 1.2

Extra node




# As it is done before, stiffness matrix of each bar:

$$K^{(1)} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^{(2)} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K^{(3)} = 40 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} = K^{(4)}$$

* Assambly global stiffness matrix:

$$
K u = \begin{bmatrix}
30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\
 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\
 & & 10 & 0 & 0 & 0 & 0 & 0 \\
 & & & 5 & 0 & -5 & 0 & 0 \\
 & SYM & & & 20 & 20 & -20 & -20 \\
 & & & & & 25 & -20 & -20 \\
 & & & & & & 40 & 40 \\
 & & & & & & & 40
\end{bmatrix}
\begin{bmatrix}
u_{x1}=0 \\ u_{y1}=0 \\ u_{x2} \\ u_{y2}=0 \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4}
\end{bmatrix}
$$

The global stiffness is singular. The last two columns and rows are a combination, therefore, the system cannot be solved.

It does not make sense to add a new node physically (so, two new displacements unknowns) because there are not additional boundary conditions.

In a truss system, the bars can only have axial deformation and the forces are applied to the nodes. This extra node does not add extra information. Before doing any calculation, it is known that the displacement of the extra node will be a linear combination of the displacements at the ends of the bar. This node would have sense if there is an extra bar between nodes 2 and 4.

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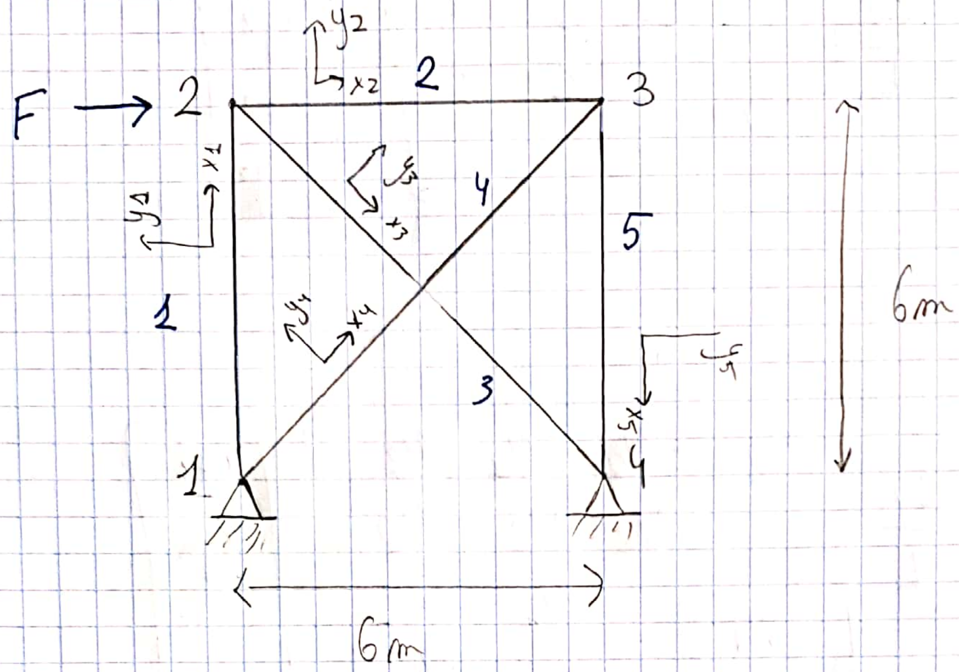
# Class problem




$L = 6\ m$

$A = 6\ cm^2$

$E = 200\ GPa$

$F = 80\ kN$

\# Stiffness method: $K^e u^e = f^e$

$$\begin{bmatrix} f_{x1} \\ f_{y2} \\ f_{x2} \\ f_{y2} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y2} \\ u_{x2} \\ u_{y2} \end{bmatrix}$$

\# Element 1

* Element 2:

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ \text{SYM} & & & 0 \end{bmatrix} \cdot \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix}$$

* Element 3

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ & 1/2 & 1/2 & -1/2 \\ & & 1/2 & -1/2 \\ \text{SYM} & & & 1/2 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix}$$

* Element 4:

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ & 1/2 & -1/2 & -1/2 \\ & & 1/2 & 1/2 \\ \text{SYM} & & & 1/2 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix}$$

* Element 5

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix}$$

For elements 1, 2, 5 $\rightarrow EA/L = 2\cdot 10^7$

For elements 3, 4 $\rightarrow EA/L = \sqrt{2}\cdot 10^7$

\# Applying global stiffness matrix:

$$K = \sqrt{2}\cdot 10^7 \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & -1/2 & -1/2 & 0 & 0 \\ & \frac{1}{2}+\sqrt{2} & 0 & -\sqrt{2} & -1/2 & -1/2 & 0 & 0 \\ & & 1/2+\sqrt{2} & -1/2 & -\sqrt{2} & 0 & -1/2 & 1/2 \\ & & & \frac{1}{2}+\sqrt{2} & 0 & 0 & 1/2 & 1/2 \\ & & & & 1/2+\sqrt{2} & 1/2 & 0 & 0 \\ & \text{SYM} & & & & 1/2+\sqrt{2} & 0 & -\sqrt{2} \\ & & & & & & 1/2 & -1/2 \\ & & & & & & & 1/2+\sqrt{2} \end{bmatrix}$$

\# Applying boundary conditions:

$u_{x1} = u_{y1} = u_{x4} = u_{y4} = 0$

$$\begin{bmatrix} 80\cdot 10^3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sqrt{2}\cdot 10^7 \begin{bmatrix} \frac{1}{2}+\sqrt{2} & -1/2 & -\sqrt{2} & 0 \\ & \frac{1}{2}+\sqrt{2} & 0 & 0 \\ & \text{SYM} & \frac{1}{2}+\sqrt{2} & 1/2 \\ & & & \frac{1}{2}+\sqrt{2} \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

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+ Solving the linear system :

$$\begin{bmatrix} u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{bmatrix} = \begin{bmatrix} 0.008541339 \text{ m} \\ 0.002231031 \text{ m} \\ 0.00677237 \text{ m} \\ -0.001768969 \text{ m} \end{bmatrix}$$

+ Reaction forces

$$\begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_4} \\ f_{y_4} \end{bmatrix} = \begin{bmatrix} -35379.38 \text{ N} \\ -80000 \text{ N} \\ -44620.62 \text{ N} \\ 80000 \text{ N} \end{bmatrix}$$