Universitat Politecnica de Catalunya

Master of Science in Computational Mechanics

Computational Structural Mechanics and Dynamics

# Assignment 1
# The Direct Stiffness Method

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# 1 Truss problem

Three bars are subject to forces $P$ and $H$ and are connected as seen on Figure 1.1. The cross-sectional area $A$ and elastic modulus $E$ are the same for all bars. The bar (2) has a length $L$, while bars (1) and (3) have length $L/cos(\alpha)$

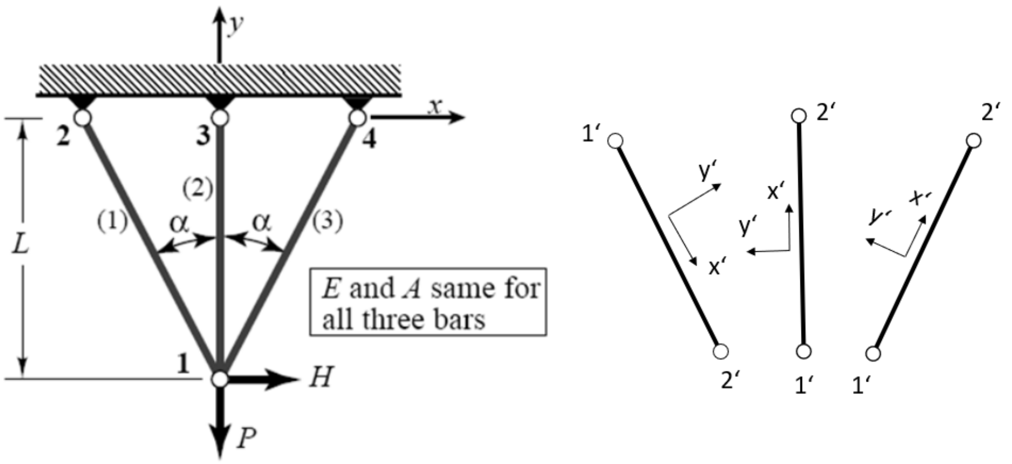


Figure 1.1: Problem geometry and discretization

## 1.1 Master stiffness equation

In order to achieve the master stiffness equation each bar must be evaluated locally within their coordinates and then transferred to the global coordinates shown on Figure 1.1. Considering the local x-axis always to be aligned with the evaluated bar (with origin on the local node 1) and taking into account Hooke's law, we can write the local system of equations relating displacements $\boldsymbol{u}$ and forces $\boldsymbol{f}$ (local variables are characterized by the upper bar):

$$\bar{\boldsymbol{K}}^{(e)}\bar{\boldsymbol{u}}^{(e)} = \frac{E^{(e)}A^{(e)}}{L^{(e)}}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{u}_{1x} \\ \bar{u}_{1y} \\ \bar{u}_{2x} \\ \bar{u}_{2y} \end{bmatrix} = \bar{\boldsymbol{f}} \tag{1.1}$$

Knowing that $\phi$ is the inclination of the bar in relation to the global x-coordinate, the coordinates transformation for the displacement vector is given by:

$$\bar{\boldsymbol{u}}^{(e)} = \boldsymbol{L}^{(e)}\boldsymbol{u}^{(e)} = \begin{bmatrix} cos(\phi^{(e)}) & sin(\phi^{(e)}) & 0 & 0 \\ -sin(\phi^{(e)}) & cos(\phi^{(e)}) & 0 & 0 \\ 0 & 0 & cos(\phi^{(e)}) & sin(\phi^{(e)}) \\ 0 & 0 & -sin(\phi^{(e)}) & cos(\phi^{(e)}) \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{jx} \\ u_{jy} \end{bmatrix} \tag{1.2}$$

where $i$ and $j$ are the global numbering of the nodes. The same procedure can be done for the force vector and, since $\boldsymbol{L}^{(e)}$ is an orthogonal matrix, equation 1.1 can be written as:

$$\boldsymbol{f} = \boldsymbol{L}^{(e)^T}\bar{\boldsymbol{K}}^{(e)}\boldsymbol{L}^{(e)}\boldsymbol{u}^{(e)} = \boldsymbol{K}^{(e)}\boldsymbol{u}^{(e)} \tag{1.3}$$

For the given problem we have $\phi^{(1)} = -(\frac{\pi}{2} - \alpha)$, $\phi^{(2)} = \frac{\pi}{2}$ and $\phi^{(3)} = \frac{\pi}{2} - \alpha$, thus, using the notation $c = cos(\alpha)$ and $s = sin(\alpha)$, the stiffness matrix in global coordinates for each bar is:

$$\boldsymbol{K}^{(1)} = \frac{EAc}{L} \begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix} \tag{1.4}$$

$$\boldsymbol{K}^{(2)} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{1.5}$$

$$\boldsymbol{K}^{(3)} = \frac{EAc}{L} \begin{bmatrix} s^2 & cs & -s^2 & -cs \\ cs & c^2 & -cs & -c^2 \\ -s^2 & -cs & s^2 & cs \\ -cs & -c^2 & cs & c^2 \end{bmatrix} \tag{1.6}$$

To assemble them into the master stiffness matrix we take into account the global node numbering displayed on Figure 1.1, yielding the global system of equations:

$$\boldsymbol{K}\boldsymbol{u} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ sym. & & & & & & & c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \tag{1.7}$$

The 5th column represent the effects of the x-coordinate of the displacement in the node 3 on the internal forces of each bar. Thus, the coefficients can only be zero, since node 3 is not connected to nodes 2 and 4 and it's connected vertically to node 1. Similarly, the 5th row, which represents the effects of the displacements in each node on the internal x-coordinate forces of node 3, must also contain only zeros.

## 1.2 Boundary conditions

The truss system is fixed on the joints 2,3 and 4, thus we can state:

$$u_{x2} = u_{y2} = u_{x3} = u_{y3} = u_{x4} = u_{y4} = 0 \tag{1.8}$$

This allows the reduction of the system of equations by eliminating the rows corresponding to displacements that are already known and the columns which would be multiplied by zero. Hence, the rows and columns from 3 to 8 are eliminated:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix} \tag{1.9}$$

## 1.3 Displacements on node 1

The system presented on 1.9 yields the displacements:

$$u_{x1} = \frac{HL}{2EAcs^2} \tag{1.10}$$

$$u_{y1} = -\frac{PL}{EA(1+2c^3)} \tag{1.11}$$

The equation found for the displacement on the $y$ direction, given by equation 1.11, satisfies the problem physically. For $\alpha \to 0$ all bars are aligned with coincident nodes offering the highest resistance for deformation. Accordingly, it's when equation 1.11 reaches its lowest value. Increasing $\alpha$ increases the displacement until its highest value for $\alpha \to \pi/2$.

The equation found for the displacement on the $x$ direction, although also meaningful for intermediate values of $\alpha$, has issues with the limit cases. This happens because it's senseless to calculate the displacement on the $x$ direction for $\alpha \to \pi/2$, since it would mean that bars (1) and (3) were infinite in length and horizontal. For $\alpha \to 0$, as stated before, the bars would be aligned and jointed only on one point on the ceiling. Thus, there would be no resistance for movement after imposing $H \neq 0$ and there could be no equilibrium. The bars would tend to rotate, therefore the solution "blows up".

## 1.4 Axial forces

The axial force $F$ on each member is given by the Equation

$$F^{(e)} = \frac{E^{(e)}A^{(e)}}{L^{(e)}} d^{(e)} \tag{1.12}$$

where $d$ is the elongation given by $d^{(e)} = \bar{u}_{jx}^{(e)} - \bar{u}_{ix}^{(e)}$.

To return to the local coordinates we use the relation $\bar{\boldsymbol{u}}^{(e)} = \boldsymbol{L}^{(e)}\boldsymbol{u}^{(e)}$, remembering $\phi^{(1)} = -(\frac{\pi}{2} - \alpha)$, $\phi^{(2)} = \frac{\pi}{2}$ and $\phi^{(3)} = \frac{\pi}{2} - \alpha$. Thus, for each bar we get:

$$\bar{\boldsymbol{u}}^{(1)} = \begin{bmatrix} \bar{u}_{x1}^{(1)} \\ \bar{u}_{y1}^{(1)} \\ \bar{u}_{x2}^{(1)} \\ \bar{u}_{y2}^{(1)} \end{bmatrix} = \begin{bmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} u_{x2=0} \\ u_{y2=0} \\ u_{x1} \\ u_{y1} \end{bmatrix} \tag{1.13}$$

$$\bar{\boldsymbol{u}}^{(2)} = \begin{bmatrix} \bar{u}_{x1}^{(2)} \\ \bar{u}_{y1}^{(2)} \\ \bar{u}_{x2}^{(2)} \\ \bar{u}_{y2}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x3} = 0 \\ u_{y3} = 0 \end{bmatrix} \tag{1.14}$$

$$\bar{\boldsymbol{u}}^{(3)} = \begin{bmatrix} \bar{u}_{x1}^{(3)} \\ \bar{u}_{y1}^{(3)} \\ \bar{u}_{x2}^{(3)} \\ \bar{u}_{y2}^{(3)} \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x4} = 0 \\ u_{y4} = 0 \end{bmatrix} \tag{1.15}$$

Consequently, the elongation for each bar is:

$$d^{(1)} = (su_{x1} - cu_{y1}) - 0 = \frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)} \tag{1.16}$$

$$d^{(2)} = 0 - u_{y1} = \frac{PL}{EA(1+2c^3)} \tag{1.17}$$

$$d^{(1)} = 0 - (su_{x1} + cu_{y1}) = -\frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)} \tag{1.18}$$

Thus, applying equation 1.12 yields:

$$F^{(1)} = \frac{H}{2s} + \frac{Pc^2}{(1+2c^3)} \tag{1.19}$$

$$F^{(2)} = \frac{P}{(1+2c^3)} \tag{1.20}$$

$$F^{(3)} = -\frac{H}{2s} + \frac{Pc^2}{(1+2c^3)} \tag{1.21}$$

We note that the solution “blows up” for the bars 1 and 3 with $\alpha \to 0$. Physically, this can be explained by the fact that the system cannot hold equilibrium for this circumstance. No axial force can compensate a force $H \neq 0$ if it's perpendicular to all bars, consequently the equilibrium solution tends to infinite, while physically the bars would start rotating.

# 2 Truss problem with extra node

A truss problem solved with 3 nodes is modified to have an extra node in the middle of a bar as shown on Figure 2.1.

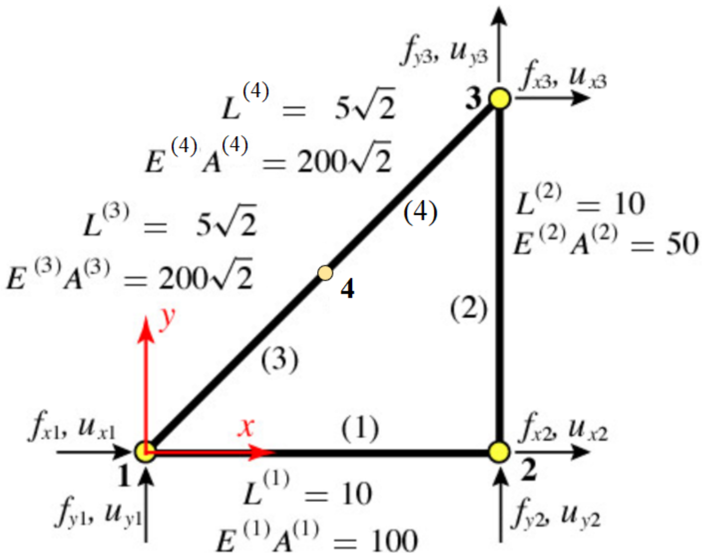


Figure 2.1: Problem geometry and discretization

Following the procedure of chapter 1, we obtain the stiffness matrix of each bar on global coordinates:

$$\boldsymbol{K}^{(1)} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.1}$$

$$\boldsymbol{K}^{(2)} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{2.2}$$

$$\boldsymbol{K}^{(3)} = 40 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \tag{2.3}$$

$$\boldsymbol{K}^{(4)} = 40 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \tag{2.4}$$

Now we can assemble the master stiffness matrix taking account the global node numbering, yielding the following system:

$$\boldsymbol{K}\boldsymbol{u} = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & & & 25 & -20 & -20 \\ & & & & & & 40 & 40 \\ sym. & & & & & & & 40 \end{bmatrix} \begin{bmatrix} u_{x1=0} \\ u_{y1=0} \\ u_{x2} \\ u_{y2=0} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \tag{2.5}$$

As we can see, the global stiffness matrix is singular. The last two rows and columns are a linear combination of themselves, thus the system cannot be solved. Physically it makes no sense adding the new node, and consequently two new displacement variables, because there are no additional boundary conditions related to it. In a truss system we consider that the bars can only have an axial deformation and that forces are applied at the nodes. With this configuration, the extra node adds no extra information, it is known *a priori* that the displacement of the extra node will be a linear combination of the displacements at the ends of the bar.