Master on Numerical
Methods in Engineering

Computational Structural Mechanics and Dynamics

Assignment 1

Direct Stiffness Method - DSM

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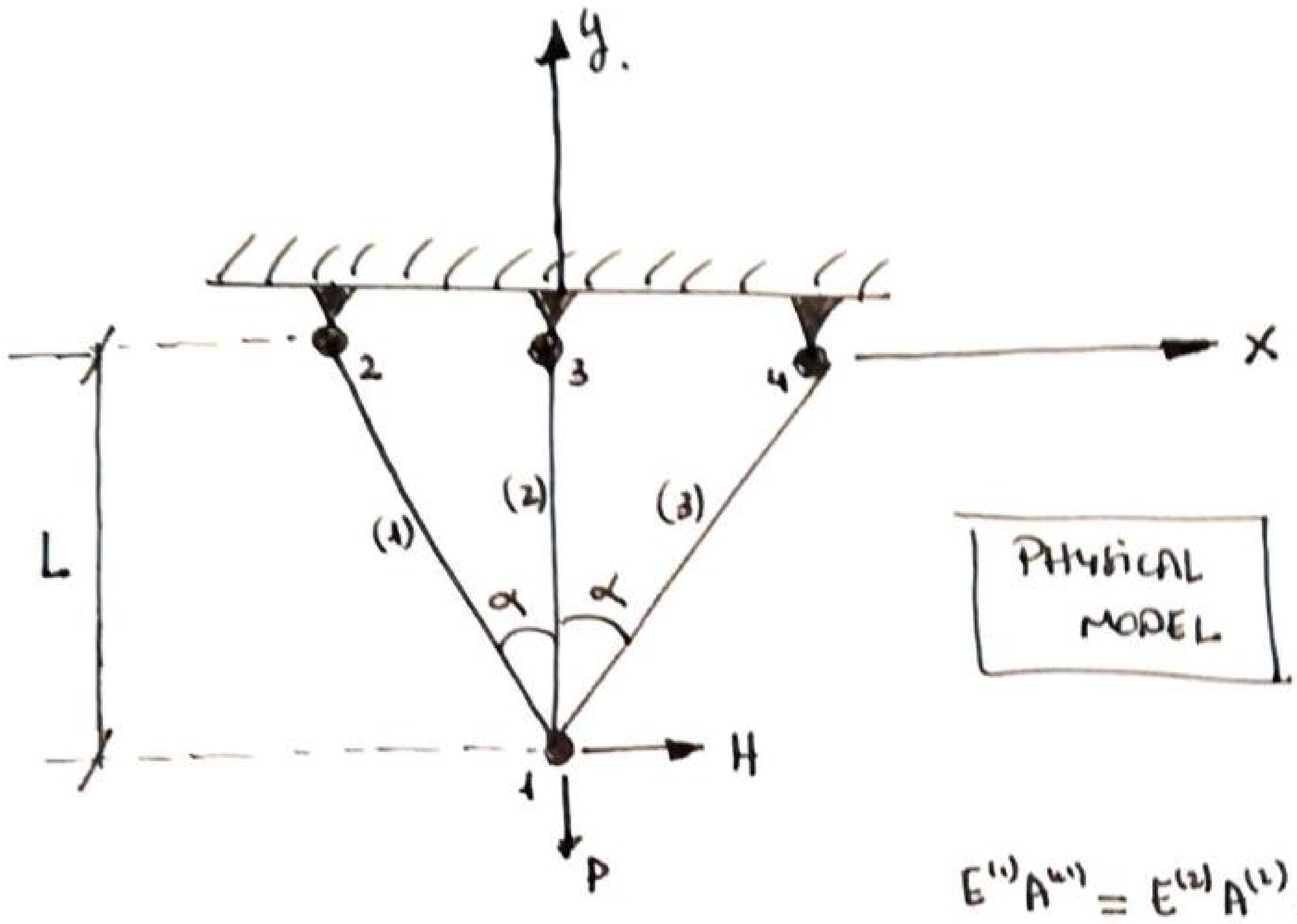
ASSIGNMENT ① (Direct Stiffness Matrix Method) DSM.

Assignment 1.1:



$E^{(1)}A^{(1)} = E^{(2)}A^{(2)} = E^{(3)}A^{(3)}$

L, α, E, A, P, H must be kept as variables

$$L^{(1)} = L^{(3)} = \frac{L}{\cos\alpha}$$

- **Degrees of freedom:**

Total 8 DOF; 6 DOF are removable by the fixed displacement conditions.

→ Fixed DOF { Node 2 → u_{x2}, u_{y2}; Node 3 → u_{x3}, u_{y3}; Node 4 → u_{x4}, u_{y4}.

→ Free to move { Node 1 → u_{x1}, u_{y1}

- **Boundary conditions:**

→ Displacement BCs: $u_{x2} = u_{y2} = u_{x3} = u_{y3} = u_{x4} = u_{y4} = 0$

→ Force BCs: $f_{x1} = H$; $f_{y1} = -P$.

1). Show that master stiffness equations are as given: Explain from physics why 5th row and column contain only zeros.

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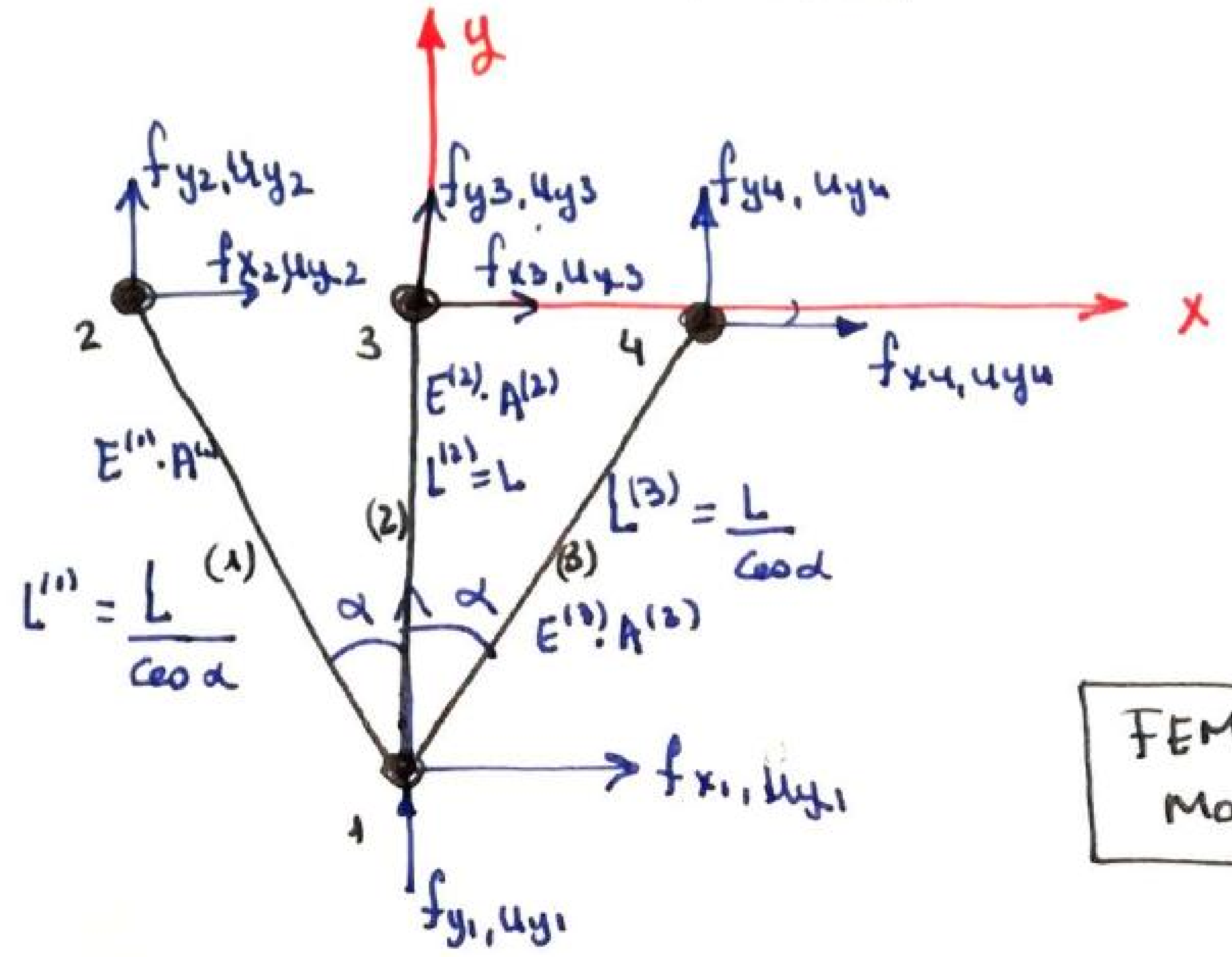


$\left.\begin{matrix} u_{xi} \\ u_{yi} \end{matrix}\right\}$ DOFs.

FEM Model

- Member (element) stiffness equations: $\bar{f} = \bar{K} \cdot \bar{u}$

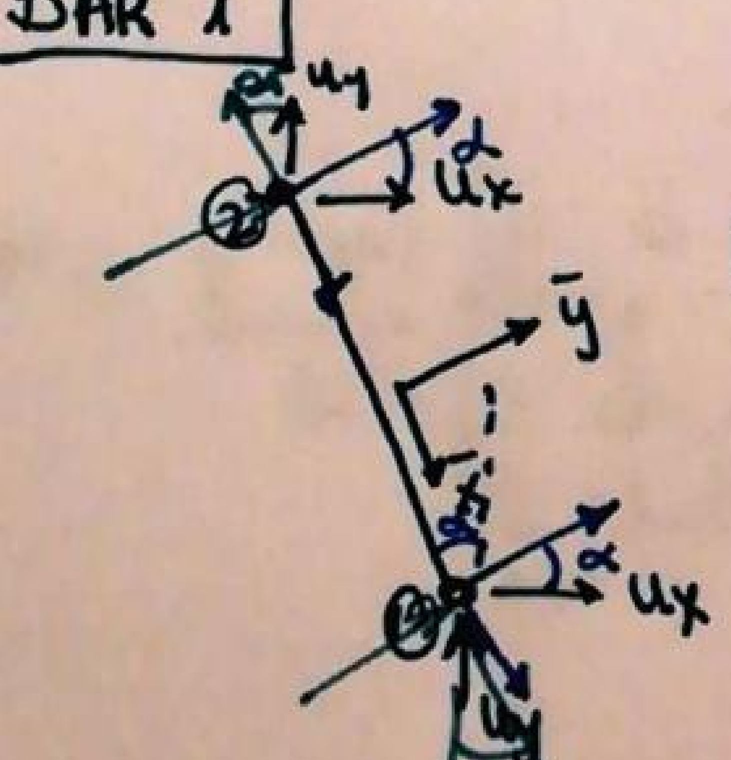
$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \frac{EA}{L} \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix}$$

Generic matrix $\bar{K} = \bar{K}^e$ to use in all the 3 bars.

- Globalization: Assembly:

→ Displacement and force transformation: Coordinate transformation.

BAR 1



$$N_1 \left\{ \begin{aligned} \bar{u}_x &= u_x \cdot s - u_y \cdot c \\ \bar{u}_y &= u_x c + u_y s \end{aligned} \right.$$

$$N_2 \left\{ \begin{aligned} \bar{u}_x &= u_x s - u_y c \\ \bar{u}_y &= u_x c + u_y s. \end{aligned} \right.$$

$$\bar{u}^{(1)} = T^e \cdot u^e \rightarrow \begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x2} \\ \bar{u}_{y2} \end{bmatrix} = \begin{bmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix} ; \quad f^e = \left[T^e\right]^T \cdot \bar{f}^e \rightarrow \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1} \\ \bar{f}_{x2} \\ \bar{f}_{y2} \end{bmatrix}$$

↑ displ. transf. matrix

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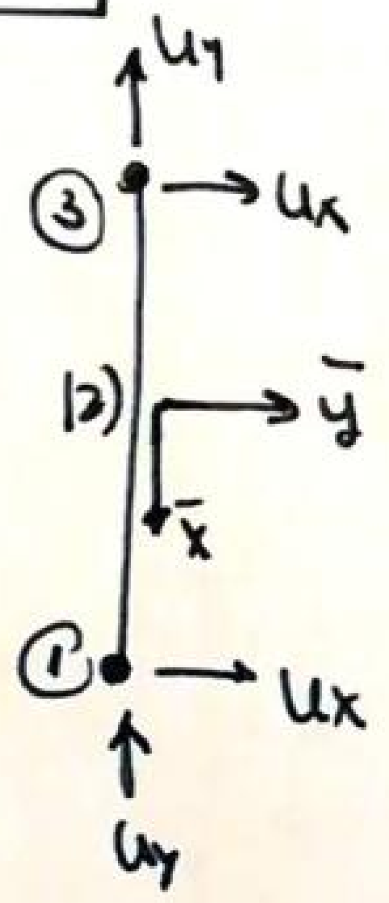
$$K^e = [T^e]^T \cdot \bar{K}^e \cdot T^e \qquad L^{(1)} = \frac{L}{\cos\alpha} = \frac{L}{c}$$

$$k^{(1)} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \frac{E^{(1)}A^{(1)}}{L^{(1)}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{bmatrix} = \frac{E^{(1)}A^{(1)}}{L} \cdot c \begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}$$

$$k^{(1)} = \frac{E^{(1)}A^{(1)}}{L} \begin{bmatrix} cs^2 & -c^2 s & -cs^2 & c^2 s \\ -c^2 s & c^3 & c^2 s & -c^3 \\ -cs^2 & c^2 s & cs^2 & -c^2 s \\ c^2 s & -c^3 & -c^2 s & c^3 \end{bmatrix}$$

sym

BAR 2



N1 $\begin{cases} \bar{u}_x = -u_y \\ \bar{u}_y = u_x \end{cases}$

N3 $\begin{cases} \bar{u}_x = -u_y \\ \bar{u}_y = u_x \end{cases}$

$$\bar{u}^{(2)} = T^{(2)} u^{(2)} \quad ; \quad T^{(2)} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

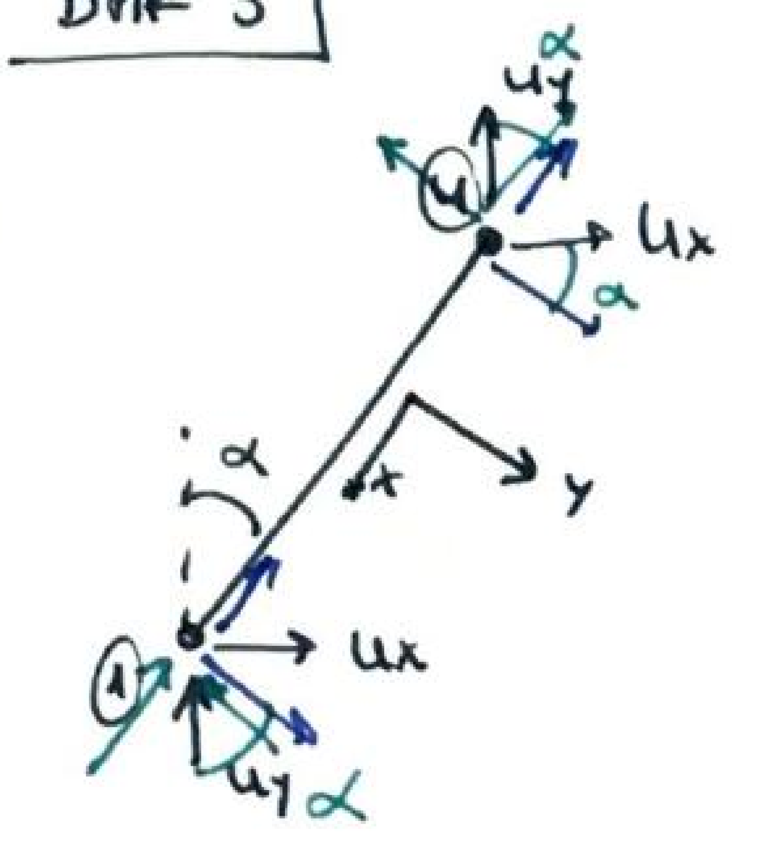
$$f^{(2)} = [T^{(2)}]^T \cdot \bar{f}^{(2)} \quad ; \quad [T^{(2)}]^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$K^{(2)} = [T^{(2)}]^T \cdot \bar{K}^{(2)} \cdot T^{(2)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \frac{E^{(2)}A^{(2)}}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$k^{(2)} = \frac{E^{(2)}A^{(2)}}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

sym

BAR 3



$$N1 \begin{cases} \bar{u}_x = -u_x s - u_y c \\ \bar{u}_y = u_x c + u_y s \end{cases}$$

$$N4 \begin{cases} \bar{u}_x = -u_x s - u_y c \\ \bar{u}_y = u_x c - u_y s \end{cases}$$

$$T^{(3)} = \begin{bmatrix} -s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & -s & -c \\ 0 & 0 & c & -s \end{bmatrix}$$

$$T^{(3)T} = \begin{bmatrix} -s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix}$$

$$K^{(3)} = \begin{bmatrix} -s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix} \frac{E^{(3)}A^{(3)}}{L^{(3)}} \cdot \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & -s & -c \\ 0 & 0 & c & -s \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} cs^2 & c^2s & -cs^2 & -c^2s \\ c^2s & c^3 & -c^2s & -c^3 \\ -cs^2 & -c^2s & cs^2 & c^2s \\ -c^2s & -c^3 & c^2s & c^3 \end{bmatrix}$$

sym.

③ Global/Master Stiffness Matrix:

$$K = \begin{bmatrix} 2c^2s & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ 0 & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ sc^2 & -c^3 & -c^2s & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & c^2s \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & c^2s & c^3 \end{bmatrix}$$

symmetric.

$c = \cos\alpha$
$s = \sin\alpha$

③ Forming the master Stiffness Equations through the Governing rules:

- Compatibility displacements
- Force equilibrium.

$$f = f^{(1)} + f^{(2)} + f^{(3)} = \left(K^{(1)} + K^{(2)} + K^{(3)}\right) \cdot u = K \cdot u.$$

$$\begin{bmatrix} \overset{H}{\cancel{f_{x1}}} \\ \overset{-P}{\cancel{f_{y1}}} \\ \cancel{f_{x2}}^{0} \\ \cancel{f_{y2}}^{0} \\ \cancel{f_{x3}}^{0} \\ \cancel{f_{y3}}^{0} \\ \cancel{f_{x4}}^{0} \\ \cancel{f_{y4}}^{0} \end{bmatrix} = \underbrace{\frac{EA}{L}}_{\text{Equal for the 3 bars.}} \cdot \begin{array}{r} \\ \\ \\ \\ \\ \text{5th row} \\ \\ \\ \\ \end{array} \begin{array}{c} \text{5th column} \\ \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & \text{SYMM.} & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix} \end{array} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

5th row & Column:

They both are related to node 3 in the x direction.
They are 0 values because force H in node 1 is assumed only on the vertical direction y.

2) Apply the BCs and show the 2-equation modified ~~stiffness~~ system

If applying BCs from page ①; the master stiffness Eqs. are reduced to:

$$\begin{bmatrix} H \\ -P \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix}$$

3) Solve for u_{x1} and u_{y1}. Check if solution makes physical sense for the limit cases $\alpha \to 0$ and $\alpha \to \pi/2$. Why does u_{x1} blow up if $H \neq 0$ and $\alpha \to 0$?

Solving the system of equations from part 2:

$$H = \frac{EA}{L} \cdot 2cs^2 \cdot u_{x1} \longrightarrow u_{x1} = \frac{HL}{EA\,2cs^2}$$

$$-P = \frac{EA}{L}\left(1+2c^3\right) u_{y1} \longrightarrow u_{y1} = -\frac{PL}{EA(1+2c^3)}$$

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- if $\alpha \to 0 \to \begin{cases} u_{x_1} = \infty \\ u_{y_1} = \frac{-PL}{3EA} \end{cases}$

It implies that the structure will be similar to 3 parallel bars.
The horizontal displacement from bars ~~with~~ will tend to ∞.

If $H \neq 0$ in the scenario of $\alpha \to 0$; due to the structure comparation it will no have resistance to the result.

- if $\alpha \to \pi/2\ (90°) \begin{cases} u_{x_1} = \infty \\ u_{y_1} = \frac{-PL}{EA} \end{cases}$

4). Recover axial forces in 3 bars. Partial answer $F^{(3)} = \frac{-H}{2S} + \frac{Pc^2}{1+2c^3}$
Why do $F^{(1)}$ and $F^{(3)}$ blow up if $H \neq 0$ and $\alpha \to 0$?

Axial force of each element: $F^{(e)} = \frac{E^e A^e}{L^e} \underbrace{(\bar{u}_{x_j}^e - \bar{u}_{x_i}^e)}_{\text{elongation of a bar. d.}}$

Bringing back the $\bar{u} = Ku$ matrices from 1) and applying BCs:

$$\bar{u}^1 = T^{(1)} u^{(1)} \longrightarrow \begin{bmatrix} u'_{x_1} \\ u'_{y_1} \end{bmatrix} = \begin{bmatrix} s & -c \\ c & s \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \end{bmatrix}$$

$$\bar{u}^{(2)} = T^{(2)} u^{(2)} \longrightarrow \begin{bmatrix} u_{x_1}^2 \\ u_{y_1}^2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \end{bmatrix}$$

$$\bar{u}^{(3)} = T^{(3)} u^{(3)} \longrightarrow \begin{bmatrix} u_{x_1}^3 \\ u_{y_1}^3 \end{bmatrix} = \begin{bmatrix} -s & -c \\ c & s \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \end{bmatrix}$$

Elongation for each bar:

$$d^{(1)} = (\bar{u}_{x_2} - \bar{u}_{x_1}) = \left[(su_{x_1} - cu_{y_1}) - 0\right] = \frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}$$

$$d^{(2)} = (\bar{u}_{x^2} - \bar{u}_{x_1})^{(2)} = 0 - u_{y_1} = \frac{PL}{EA(1+2c^3)}$$

$$d^{(3)} = (u_{x_2} - u_{x_1})^{(3)} = 0 - (su_{x_1} + cu_{y_1}) = -\frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}$$

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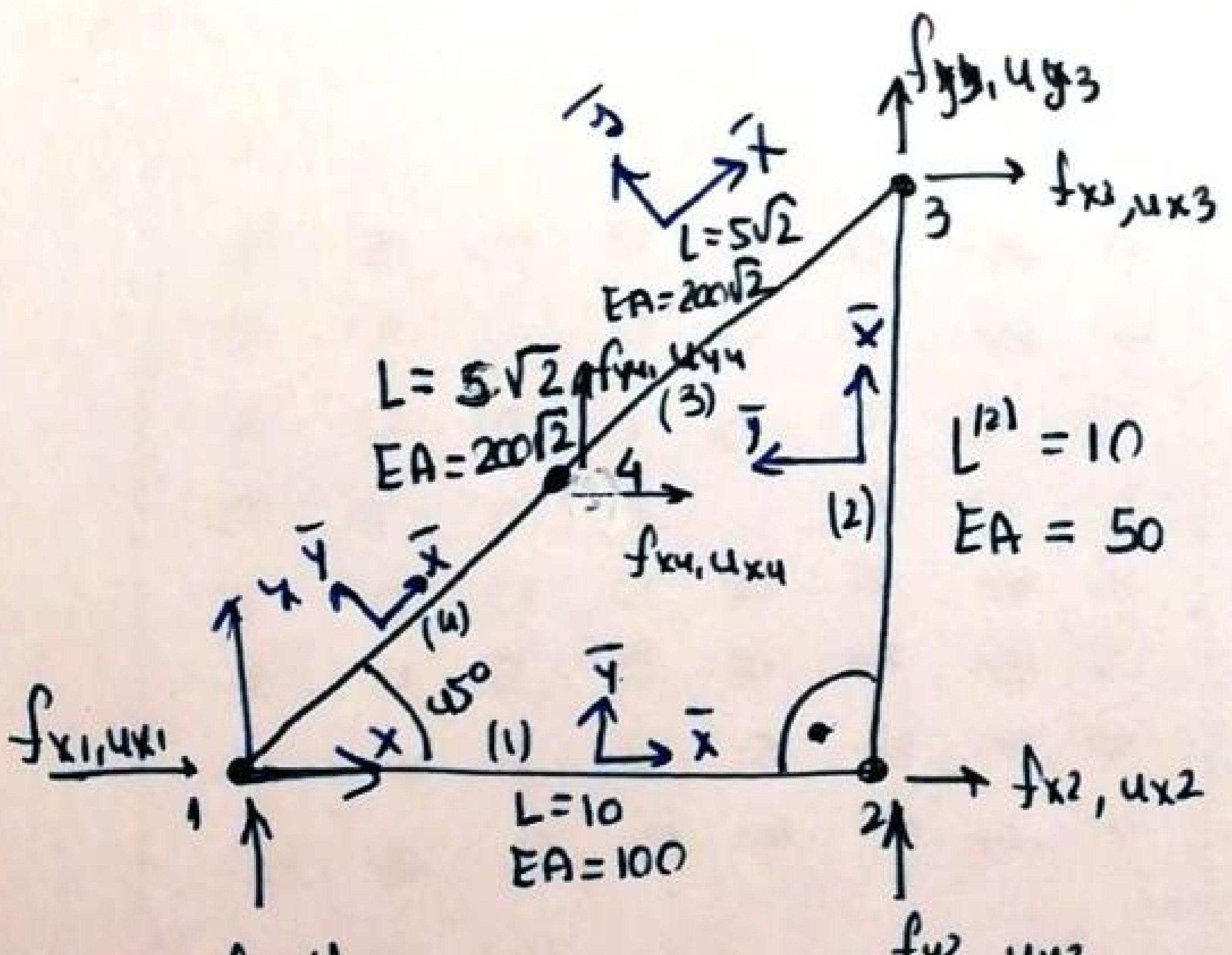
Resulting axial forces are:

$$\boxed{\begin{aligned} F^{(1)} &= \frac{H}{2S} + \frac{Pc^2}{1+2c^3} \\ F^{(2)} &= \frac{P}{1+2c^3} \\ F^{(3)} &= -\frac{H}{2S} + \frac{Pc^2}{1+2c^3} \end{aligned}}$$

Assignment 1.2:

Dr. Who. Improvement of results by extra node, 4 at the midpoint of member (3).
Reason: more is better.

Try Dr. who suggestion by hand computations and verify that solution blows up because the modified master stiffness matrix is singular. Explain.



It will be followed the same procedure as for assignment 1.1. Here only relevant results will be shown:

Bar 1

$$K^{(1)} = \frac{100}{10}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = \begin{bmatrix} 10 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix}$$

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Bar 2

$$K^{(2)} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 5 \end{bmatrix}$$

Bar 3:

$$K^{(3)} = \cancel{\frac{EA}{L}}^{\,40}\begin{bmatrix} 0'5 & 0'5 & -0'5 & -0'5 \\ 0'5 & 0'5 & -0'5 & -0'5 \\ -0'5 & -0'5 & 0'5 & 0'5 \\ -0'5 & -0'5 & 0'5 & 0'5 \end{bmatrix}$$

Bar 4

$$K^{(4)} = \begin{bmatrix} 20 & 20 & -20 & -20 \\ 20 & 20 & -20 & -20 \\ -20 & -20 & 20 & 20 \\ -20 & -20 & 20 & 20 \end{bmatrix} = K^{(3)}$$

3 Global matrix:

$$K = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & & 0 & 0 & 0 & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & -5 & 20 & 25 & -20 & -20 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \end{bmatrix}$$

4 Master stiffness Equations:

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ \cancel{f_{x2}}^{\,0} \\ f_{y2} \\ \cancel{f_{x3}}^{\,2} \\ \cancel{f_{y3}}^{\,1} \\ f_{x4} \\ f_{y4} \end{bmatrix} = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & & & 25 & -20 & -20 \\ \text{Symm.} & & & & & & 40 & 40 \\ & & & & & & & 40 \end{bmatrix} \cdot \begin{bmatrix} \cancel{u_{x1}}^{\,0} \\ \cancel{u_{y1}}^{\,0} \\ u_{x2} \\ \cancel{u_{y2}}^{\,0} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Applying BCs. $\begin{cases} u_{x1} = u_{y1} = u_{y2} = 0 \\ f_{x2} = 0;\ f_{x3} = 2,\ f_{y3} = 1. \end{cases}$

The stiffness matrix is yet singular. It is required to provide more BCs in order to solve the problem.