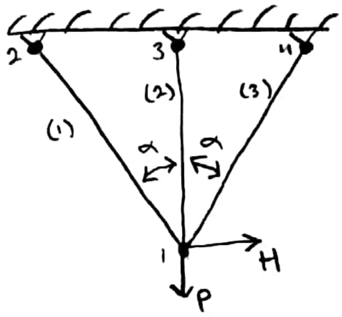
Alexander Keiser CSMD Assignment #1



the general form of the elemental stiffness matrix is

$$K^e = \frac{EA^e}{L^e}\begin{bmatrix} \cos^2(\varphi) & \sin(\varphi)\cos(\varphi) & -\cos^2(\varphi) & -\sin(\varphi)\cos(\varphi) \\ & \sin^2(\varphi) & -\sin(\varphi)\cos(\varphi) & -\sin^2(\varphi) \\ & & \cos^2(\varphi) & \sin(\varphi)\cos(\varphi) \\ \text{symmetric} & & & \sin^2(\varphi) \end{bmatrix}$$

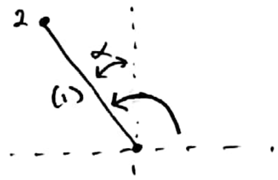
PART A

Element 1

$\varphi = \frac{\pi}{2} + \alpha$ E & A constants $L = \frac{L}{c}$

$\sin(\varphi) = \sin\left(\frac{\pi}{2}+\alpha\right) = \cos(\alpha) = c$

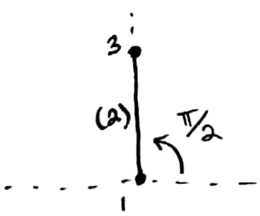
$\cos(\varphi) = \cos\left(\frac{\pi}{2}+\alpha\right) = -\sin(\alpha) = -s$



$$K^1 = \frac{EA}{L}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} cs^2 & -sc^2 & -cs^2 & sc^2 \\ & c^3 & sc^2 & -c^3 \\ & & cs^2 & -sc^2 \\ \text{symmetric} & & & c^3 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Element 2

$\varphi = \frac{\pi}{2}$ E, A, & L constants $\sin(\varphi) = 1$ $\cos(\varphi) = 0$



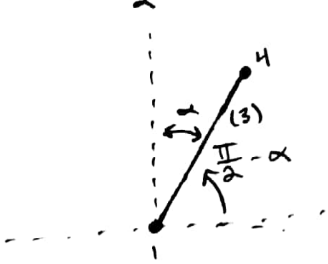
$$K^2 = \frac{EA}{L}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ \text{symmetric} & & & 1 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$

Element 3

$\varphi = \frac{\pi}{2} - \alpha$ E & A constants $L = \frac{L}{c}$

$\sin(\varphi) = \sin\left(\frac{\pi}{2}-\alpha\right) = \cos(\alpha) = c$

$\cos(\varphi) = \cos\left(\frac{\pi}{2}-\alpha\right) = \sin(\alpha) = s$



$$K^3 = \frac{EA}{L}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \\ \begin{bmatrix} cs^2 & sc^2 & -cs^2 & -sc^2 \\ & c^3 & -sc^2 & -c^3 \\ & & cs^2 & sc^2 \\ \text{symmetric} & & & c^3 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \end{array}$$

Force and Displacement vectors

$\vec{F} = [F_{x1}\ F_{y1}\ F_{x2}\ F_{y2}\ F_{x3}\ F_{y3}\ F_{x4}\ F_{y4}]^T$

$\vec{U} = [U_{x1}\ U_{y1}\ U_{x2}\ U_{y2}\ U_{x3}\ U_{y3}\ U_{x4}\ U_{y4}]^T$

Boundary Conditions

$F_{x1} = H$ $F_{y1} = -P$

$F_{x2}, F_{y2}, F_{x3}, F_{y3}, F_{x4}, F_{y4} = 0$

$U_{x2}, U_{y2}, U_{x3}, U_{y3}, U_{x4}, U_{y4} = 0$

$$\begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & sc^2 & 0 & 0 & -cs^2 & -sc^2 \\ & 1+2c^3 & sc^2 & -c^3 & 0 & -1 & -sc^2 & -c^3 \\ & & cs^2 & -sc^2 & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & \text{SYMMETRIC} & & & & & cs^2 & sc^2 \\ & & & & & & & c^3 \end{bmatrix} \begin{bmatrix} U_{x1} \\ U_{y1} \\ U_{x2} \\ U_{y2} \\ U_{x3} \\ U_{y3} \\ U_{x4} \\ U_{y4} \end{bmatrix}$$

$K_{global} = K^1 + K^2 + K^3$

These results confirm with what is given in the problem statement

☆★ The 5th row & column have to be zero because the matrix is symmetric and it is not possible to have any forces in the x-direction at node 3 because the element is vertical and node 3 is not part of another element

PART B

Applying the boundary conditions will give us the following reduced system...

$$\left(\frac{EA}{L}\right)(2cs^2)^*(U_{x1}) = H$$
$$\left(\frac{EA}{L}\right)(1+2c^3)^*(U_{y1}) = -P$$

or

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} U_{x1} \\ U_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

PART C

Rearranging the above equations allows us to solve for U_{x1} & U_{y1} in terms of the rest of the variables

$$U_{x1} = \frac{H}{\left(\frac{EA}{L}\right)(2cs^2)}$$

$$U_{y1} = \frac{-P}{\left(\frac{EA}{L}\right)(1+2c^3)}$$

We will now check the limit cases on the next page →

for $\alpha \to 0$ & $H \neq 0$

$$U_{x1} = \frac{H}{\left(\frac{EA}{L}\right)\left(2\cos(0)\sin^2(0)\right)} = \frac{H}{\left(\frac{EA}{L}\right)(2 \cdot 1 \cdot 0^2)} = \frac{H}{0}$$

we can see that this answer "blows up" because U_{x1} is proportional to $1/\sin^2(\alpha)$ which in this case $(\alpha = 0)$ lends us with a zero in the denominator

$$U_{y1} = \frac{-P}{\left(\frac{EA}{L}\right)\left(1 + 2\cos^3(0)\right)} = \frac{-PL}{3EA}$$

for $\alpha \to \frac{\pi}{2}$ & $H \neq 0$

$$U_{x1} = \frac{H}{\left(\frac{EA}{L}\right)\left(2\cos(\pi/2)\sin^2(\pi/2)\right)} = \frac{H}{\left(\frac{EA}{L}\right)(2 \cdot 0 \cdot 1^2)} = \frac{H}{0}$$

This answer also makes physical sense by blowing up because the geometry is impossible & L becomes large

$$U_{y1} = \frac{-P}{\left(\frac{EA}{L}\right)\left(1 + 2\cos^3(\pi/2)\right)} = \frac{-P}{\left(\frac{EA}{L}\right)} = \frac{-PL}{EA}$$

PART D

Recovery of axial forces within the members

Element 1

$$u^{(1)} = [U_{x1}\ U_{y1}\ U_{x2}\ U_{y2}]^T$$

$$u^{(1)} = \left[\frac{HL}{2EAcs^2}\ \ \frac{-PL}{EA(1+2c^3)}\ \ 0\ \ 0\right]^T$$

$\sin(\varphi) = c \quad \cos(\varphi) = -s$

$$T = \begin{bmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix}$$

$\varphi = \frac{\pi}{2} + \alpha$

converting to local coordinates

$$\begin{bmatrix} \bar{U}_{x1} \\ \bar{U}_{y1} \\ \bar{U}_{x2} \\ \bar{U}_{y2} \end{bmatrix} = \begin{bmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix} \begin{bmatrix} \frac{HL}{2EAcs^2} \\ \frac{-PL}{EA(1+2c^3)} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-HL}{2EAcs} - \frac{PLc}{EA(1+2c^3)} \\ \frac{-HL}{2EAs^2} + \frac{PLs}{EA(1+2c^3)} \\ 0 \\ 0 \end{bmatrix}$$

$$d^{(1)} = \bar{U}_{x2} - \bar{U}_{x1} = 0 - \left(\frac{-HL}{2EAcs} - \frac{PLc}{EA(1+2c^3)}\right) = \frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}$$

$L^{(1)} = \frac{L}{c}$

$$F^{(1)} = \frac{EA}{L^{(1)}} d^{(1)} = \frac{EAc}{\cancel{L}}\left(\frac{H\cancel{L}}{2\cancel{EA}cs} + \frac{P\cancel{L}c}{\cancel{EA}(1+2c^3)}\right)$$

$$\boxed{F^{(1)} = \frac{H}{2s} + \frac{Pc^2}{1+2c^3}}$$

Element 2

$$U^{(2)} = \left[U_{x1} \quad U_{y1} \quad U_{x3} \quad U_{y3} \right]^T \qquad \varphi = \frac{\pi}{2} \qquad \sin(\varphi) = 1 \qquad \cos(\varphi) = 0$$

$$U^{(2)} = \left[\frac{HL}{2EAcs^2} \quad \frac{-PL}{EA(1+2c^3)} \quad 0 \quad 0 \right]^T \qquad T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \bar{U}_{x1} \\ \bar{U}_{y1} \\ \bar{U}_{x3} \\ \bar{U}_{y3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{HL}{2EAcs^2} \\ \frac{-PL}{EA(1+2c^3)} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-PL}{EA(1+2c^3)} \\ \frac{-HL}{2EAcs^2} \\ 0 \\ 0 \end{bmatrix}$$

$$d^{(2)} = \bar{U}_{x3} - \bar{U}_{x1} = 0 - \left(\frac{-PL}{EA(1+2c^3)} \right) = \frac{PL}{EA(1+2c^3)}$$

$$F^{(2)} = \frac{EA}{L^{(2)}} d^{(2)} = \frac{EA}{\cancel{L}} \left(\frac{P\cancel{L}}{EA(1+2c^3)} \right) = \frac{P}{1+2c^3}$$

$$\boxed{F^{(2)} = \frac{P}{1+2c^3}}$$

Element 3

$$U^{(3)} = \left[U_{x1} \quad U_{y1} \quad U_{x4} \quad U_{y4} \right]^T \qquad \varphi = \frac{\pi}{2} - \alpha \qquad \sin(\varphi) = c \qquad \cos(\varphi) = s$$

$$U^{(3)} = \left[\frac{HL}{2EAcs^2} \quad \frac{-PL}{EA(1+2c^3)} \quad 0 \quad 0 \right]^T \qquad T = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix}$$

$$\begin{bmatrix} \bar{U}_{x1} \\ \bar{U}_{y1} \\ \bar{U}_{x4} \\ \bar{U}_{x4} \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} \frac{HL}{2EAcs^2} \\ \frac{-PL}{EA(1+2c^3)} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{HL}{2EAcs} - \frac{PLc}{EA(1+2c^3)} \\ \frac{HL}{2EAs^2} - \frac{-PLs}{EA(1+2c^3)} \\ 0 \\ 0 \end{bmatrix}$$

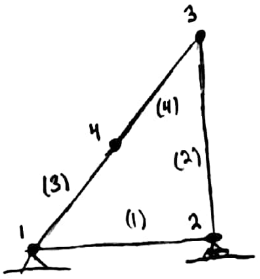
$$d^{(3)} = \bar{U}_{x4} - \bar{U}_{x1} = 0 - \left(\frac{HL}{2EAcs} - \frac{PLc}{EA(1+2c^3)} \right)$$

$$F^{(3)} = \frac{EA}{L^{(3)}} d^{(3)} = \frac{EA\cancel{c}}{\cancel{L}} \left(\frac{-H\cancel{L}}{2\cancel{EA}\cancel{c}s} + \frac{PLc}{EA(1+2c^3)} \right)$$

$$\boxed{F^{(3)} = \frac{-H}{2s} + \frac{Pc^2}{1+2c^3}}$$

$F^{(1)}$ & $F^{(3)}$ will "blow up" if $H \neq 0$ and $\alpha \to 0$ because the first term of both $F^{(1)}$ & $F^{(3)}$ are inversley proportional to $\sin(\alpha)$ and $\sin(0) = 0$, this causes the forces to "blow up" in the direction of infinity.

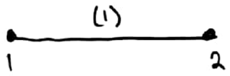
Assignment 2



The general form of the elemental stiffness matrix is

$$K^e = \frac{E^e A^e}{L^e} \begin{bmatrix} \cos^2(\varphi) & \sin(\varphi)\cos(\varphi) & -\cos^2(\varphi) & -\sin(\varphi)\cos(\varphi) \\ & \sin^2(\varphi) & -\sin(\varphi)\cos(\varphi) & -\sin^2(\varphi) \\ & & \cos^2(\varphi) & \sin(\varphi)\cos(\varphi) \\ \underbrace{\text{symmetric}} & & & \sin^2(\varphi) \end{bmatrix}$$

Element 1



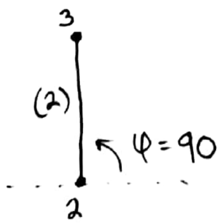
$\varphi = 0$ $\quad \frac{E^1 A^1}{L^1} = 10$

$\sin(\varphi) = \sin(0) = 0$

$\cos(\varphi) = \cos(0) = 1$

$$K^1 = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ \underbrace{\text{symmetric}} & & & 0 \end{bmatrix}$$

Element 2



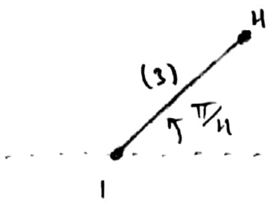
$\frac{E^2 A^2}{L^2} = 5$

$\sin(\varphi) = \sin(\frac{\pi}{2}) = 1$

$\cos(\varphi) = \cos(\frac{\pi}{2}) = 0$

$$K^2 = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ \underbrace{\text{symmetric}} & & & 1 \end{bmatrix}$$

Element 3



$\varphi = \frac{\pi}{4}$

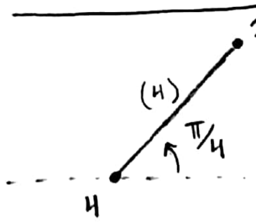
$$\frac{E^3A^3}{L^3} = \frac{200\cancel{\sqrt{2}}}{5\cancel{\sqrt{2}}} = 40$$

$\sin(\varphi) = \sin(\pi/4) = .707$

$\cos(\varphi) = \cos(\pi/4) = .707$

$$K^3 = 40 \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}$$

Element 4



$\varphi = \pi/4$

$$\frac{E^4A^4}{L^4} = \frac{200\sqrt{2}}{5\sqrt{2}} = 40$$

$\sin(\varphi) = \sin(\pi/4) = .707$

$\cos(\varphi) = \cos(\pi/4) = .707$

$$K^4 = 40 \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}$$

Force & Displacement vectors

$\vec{F} = [F_{x1}\ F_{y1}\ F_{x2}\ F_{y2}\ F_{x3}\ F_{y3}\ F_{x4}\ F_{y4}]^T$

$\vec{U} = [U_{x1}\ U_{y1}\ U_{x2}\ U_{y2}\ U_{x3}\ U_{y3}\ U_{x4}\ U_{y4}]^T$

Boundary Conditions

$U_{x1} = U_{y1} = U_{y2} = 0$

$F_{x3} = 2 \quad F_{y3} = 1$

we will now assemble the Matrix equations with Boundary Conditions

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & -5 & 20 & 25 & -20 & -20 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ U_{2x} \\ 0 \\ U_{3x} \\ U_{3y} \\ U_{4x} \\ U_{4y} \end{bmatrix}$$

we will now reduce the system and attempt to solve for the displacements, the system reduces to a 5x5

$$\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & -20 & -20 \\ 0 & 20 & 25 & -20 & -20 \\ 0 & -20 & -20 & 40 & 40 \\ 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} U_{2x} \\ U_{3x} \\ U_{3y} \\ U_{4x} \\ U_{4y} \end{bmatrix}$$

EQUAL ROWS

Since we have 2 rows that are equal, we know that the determinant of the matrix must be zero. If the Determinant is zero, we know the matrix is singular. The above system has infinite solutions as a result and the solution "Blows up". The physical meaning behind this is because the mathematical model of the system has no built in way of resisting loads applied at this point, making the problem not solvable. We could fix by adding another diagonal element from node 2 to node 4. This is shown in the sketch here →