Homework 1: The Direct Stiffness Method

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1.1.1:

Given:

$$P, H, L, E, A, \alpha$$

BC:

$$u_{x_2} = u_{x_3} = u_{x_4} = 0$$
$$u_{y_2} = u_{y_3} = u_{y_4} = 0$$

$$c = cos\alpha, s = sin\alpha$$

Searched:

Show that the master stiffness equations are

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ symm & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

explain from physics why the 5th row and column contains only zeros.

Solution:

In a truss member the following relationship is true

$$F^{(e)} = k_s^{(e)} d^{(e)} = \frac{E^{(e)}A^{(e)}}{L^{(e)}} d^{(e)},$$

$$F^{(e)} = \bar{f}_{xj}^{(e)} = -\bar{f}_{xi}^{(e)},$$
$$d^{(e)} = \bar{u}_{x_j}^{(e)} - \bar{u}_{x_i}^{(e)},$$

which concluded in matrix form becomes

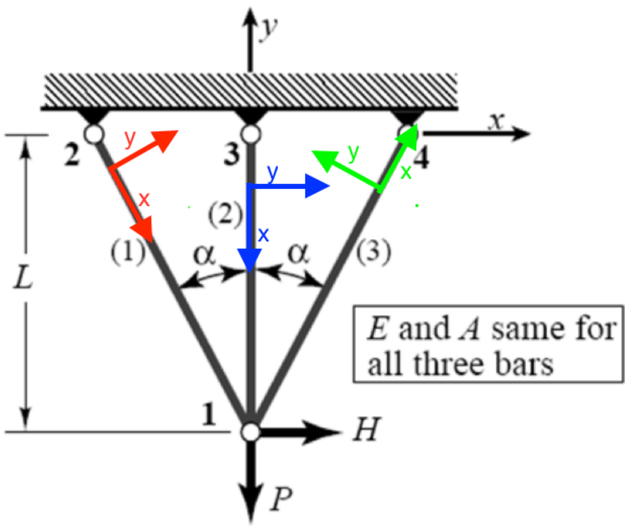
$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}.$$

This results in the local stiffness matrices

$$\overline{\boldsymbol{K}}^{(1)} = \overline{\boldsymbol{K}}^{(3)} = \frac{EA}{L/c}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{\boldsymbol{K}}^{(2)} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Choosing local coordinates the local x-axis should be aligned with the truss member. To make the problem as simple as possible it is also recommended to choose the directions of the local x- and y-axis yielding the smallest possible angle of deviation from the global x- and y-axis. The following local coordinates are chosen for the three members.



The chosen local coordinates result in relationships between local and global displacements and forces. Using

$$c = cos\alpha, \qquad s = sin\alpha$$

the displacement relationship for member (1) becomes

$$\begin{bmatrix} \bar{u}_{x1}^{(1)} \\ \bar{u}_{y1}^{(1)} \\ \bar{u}_{x2}^{(1)} \\ \bar{u}_{y2}^{(1)} \end{bmatrix} = \begin{bmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{bmatrix}\begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix}$$

$$\overline{\boldsymbol{u}}^{(1)} = \boldsymbol{T}^{(1)}\boldsymbol{u}^{(1)}$$

and the force relationship

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} \bar{f}_{x1}^{(1)} \\ \bar{f}_{y1}^{(1)} \\ \bar{f}_{x2}^{(1)} \\ \bar{f}_{y2}^{(1)} \end{bmatrix}.$$

For member (2) the relationships become

$$\begin{bmatrix} \bar{u}_{x1}^{(2)} \\ \bar{u}_{y1}^{(2)} \\ \bar{u}_{x3}^{(2)} \\ \bar{u}_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_{x1}^{(2)} \\ u_{y1}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix},$$

$$\bar{\boldsymbol{u}}^{(2)} = \boldsymbol{T}^{(2)} \boldsymbol{u}^{(2)},$$

$$\begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{f}_{x1}^{(2)} \\ \bar{f}_{y1}^{(2)} \\ \bar{f}_{x3}^{(2)} \\ \bar{f}_{y3}^{(2)} \end{bmatrix}.$$

Lastly for member (3) the relationships become

$$\begin{bmatrix} \bar{u}_{x1}^{(3)} \\ \bar{u}_{y1}^{(3)} \\ \bar{u}_{x4}^{(3)} \\ \bar{u}_{y4}^{(3)} \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x4}^{(3)} \\ u_{y4}^{(3)} \end{bmatrix},$$

$$\bar{\boldsymbol{u}}^{(3)} = \boldsymbol{T}^{(3)} \boldsymbol{u}^{(3)},$$

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x4}^{(3)} \\ f_{y4}^{(3)} \end{bmatrix} = \begin{bmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} \bar{f}_{x1}^{(3)} \\ \bar{f}_{y1}^{(3)} \\ \bar{f}_{x4}^{(3)} \\ \bar{f}_{y4}^{(3)} \end{bmatrix}.$$

Using the relationships

$$\bar{\boldsymbol{f}}^{(e)} = \boldsymbol{T}^{(e)} \boldsymbol{f}^{(e)},$$
$$\bar{\boldsymbol{u}}^{(e)} = \boldsymbol{T}^{(e)} \boldsymbol{u}^{(e)}$$
$$\bar{\boldsymbol{K}}^{(e)} \bar{\boldsymbol{u}}^{(e)} = \bar{\boldsymbol{f}}^{(e)}$$

the global stiffness matrix can be expressed in terms of the local stiffness matrix and the transformation matrix as follows:

$$\bar{\boldsymbol{K}}^{(e)} \boldsymbol{T}^{(e)} \boldsymbol{u}^{(e)} = \boldsymbol{T}^{(e)} \boldsymbol{f}^{(e)}$$
$$\left(\boldsymbol{T}^{(e)}\right)^T \bar{\boldsymbol{K}}^{(e)} \boldsymbol{T}^{(e)} \boldsymbol{u}^{(e)} = \boldsymbol{f}^{(e)}$$

$$\boldsymbol{K}^{(e)}\boldsymbol{u}^{(e)} = \boldsymbol{f}^{(e)}$$

$$\boldsymbol{K}^{(e)} = \left(\boldsymbol{T}^{(e)}\right)^T \overline{\boldsymbol{K}}^{(e)}\boldsymbol{T}^{(e)}.$$

Using the calculated transformation and stiffness matrices the following global stiffness matrices are recieved:

$$\boldsymbol{K}^{(1)} = \left(\boldsymbol{T}^{(1)}\right)^T \overline{\boldsymbol{K}}^{(1)}\boldsymbol{T}^{(1)} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \frac{EA}{L/c} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{bmatrix} =$$
$$= \frac{EAc}{L} \begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -cs & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix}$$

$$\boldsymbol{K}^{(2)} = \left(\boldsymbol{T}^{(2)}\right)^T \overline{\boldsymbol{K}}^{(2)}\boldsymbol{T}^{(2)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} =$$
$$= \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{K}^{(3)} = \left(\boldsymbol{T}^{(3)}\right)^T \overline{\boldsymbol{K}}^{(3)}\boldsymbol{T}^{(3)} = \begin{bmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{bmatrix} \frac{EA}{L/c} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} =$$

$$= \frac{EAc}{L} \begin{bmatrix} s^2 & sc & -s^2 & -sc \\ sc & c^2 & -sc & -c^2 \\ -s^2 & -sc & s^2 & sc \\ -sc & -c^2 & sc & c^2 \end{bmatrix}.$$

For a hand written assembly the stiffness matrices are expanded to make the process easier. The expanded matrices result in the following systems for member (1), (2) and (3):

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \\ f_{x4}^{(1)} \\ f_{y4}^{(1)} \end{bmatrix} = \frac{EAc}{L}\begin{bmatrix} s^2 & -sc & -s^2 & sc & 0 & 0 & 0 & 0 \\ -sc & c^2 & sc & -c^2 & 0 & 0 & 0 & 0 \\ -s^2 & sc & s^2 & -sc & 0 & 0 & 0 & 0 \\ sc & -c^2 & -sc & c^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \\ u_{x3}^{(1)} \\ u_{y3}^{(1)} \\ u_{x4}^{(1)} \\ u_{y4}^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \\ f_{x4}^{(2)} \\ f_{y4}^{(2)} \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{x1}^{(2)} \\ u_{y1}^{(2)} \\ u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \\ u_{x4}^{(2)} \\ u_{y4}^{(2)} \end{bmatrix}$$

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x2}^{(3)} \\ f_{y2}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \\ f_{x4}^{(3)} \\ f_{y4}^{(3)} \end{bmatrix} = \frac{EAc}{L}\begin{bmatrix} s^2 & sc & 0 & 0 & 0 & 0 & -s^2 & -sc \\ sc & c^2 & 0 & 0 & 0 & 0 & -sc & -c^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -s^2 & -sc & 0 & 0 & 0 & 0 & s^2 & sc \\ -sc & -c^2 & 0 & 0 & 0 & 0 & sc & c^2 \end{bmatrix}\begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x2}^{(3)} \\ u_{y2}^{(3)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \\ u_{x4}^{(3)} \\ u_{y4}^{(3)} \end{bmatrix}.$$

The local displacements $u_{x_i}^{(e)}$ can be written in global form using the compability

$$u_{x_i}^{(1)} = u_{x_i}^{(2)} = u_{x_i}^{(3)} = u_{x_i}$$
$$u_{y_i}^{(1)} = u_{y_i}^{(2)} = u_{y_i}^{(3)} = u_{y_i}$$

resulting in the master stiffness equations

$$
\begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & sc^2 & 0 & 0 & -s^2c & -sc^2 \\ 0 & 1+2c^3 & sc^2 & -c^3 & 0 & -1 & -sc^2 & -c^3 \\ -s^2c & sc^2 & s^2c & -sc^2 & 0 & 0 & 0 & 0 \\ sc^2 & -c^3 & -sc^2 & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -s^2c & -sc^2 & 0 & 0 & 0 & 0 & s^2c & sc^2 \\ -sc^2 & -c^3 & 0 & 0 & 0 & 0 & sc^2 & c^3 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}.
$$

Using $f_{x_1} = H, f_{y_1} = -P$ and zeros for the rest it holds that the master stiffness equation is

$$
\frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & sc^2 & 0 & 0 & -s^2c & -sc^2 \\ & 1+2c^3 & sc^2 & -c^3 & 0 & -1 & -sc^2 & -c^3 \\ & & s^2c & -sc^2 & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & s^2c & sc^2 \\ symm & & & & & & & c^3 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The fifth column represents the effect the displacement in x-direction of node 3 has on the internal forces on the bar. The coefficients are therefore zero since bar (2) is vertical and node 3 is not horisontally connected to neither node 2 or 4. The fift row represents the effect from other displacements on the internal forces at node 3 in the x-direction which by the same arguments also should be zero.

1.1.2:

Searched:

Apply boundary conditions and show the 2-equation modified stiffness system.

Solution:

Applying the boundary conditions

$$
u_{x_2} = u_{x_3} = u_{x_4} = 0
$$
$$
u_{y_2} = u_{y_3} = u_{y_4} = 0
$$

the system is reduced to

$$
\frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}.
$$

1.1.3:

Searched:

Solve for u_{x_1}, u_{y_1}. Check that makes sense for $\alpha \to 0, \alpha \to \frac{\pi}{2}$.

Why u_{x_1} ”blow up” for $\alpha \to 0, H \neq 0$?

Solution:

The system

$$\frac{EA}{L}\begin{bmatrix}2cs^2 & 0\\ 0 & 1+2c^3\end{bmatrix}\begin{bmatrix}u_{x_1}\\ u_{y_1}\end{bmatrix}=\begin{bmatrix}H\\ -P\end{bmatrix}$$

result in the displacements

$$u_{x_1}=\frac{HL}{2EAcs^2}$$
$$u_{y_1}=-\frac{PL}{EA(1+2c^3)}.$$

For the limiting cases $\alpha\to 0$ and $\alpha\to\frac{\pi}{2}$ u_{y_1} takes on its highest and lowest values. This makes physical sense for $\alpha\to 0$ since in this case all three bars are vertical hence maximum resistance to displacement in y direction. For $\alpha\to\frac{\pi}{2}$ bar (1) and (3) would be infinitely large and therefore approximately be at a horisontal position yielding only the middle bar (2) giving the vertical displacement resistance.

The displacement u_{x_1} does not handle the limiting cases well due to a resulting division with zero. Note however that the limiting case $\alpha\to\frac{\pi}{2}$ isn't physical since this would mean bars (1) and (3) having infinite length. For $\alpha\to 0$ the problem would be one bar with free rotation around point (3). As a consequence there would be no resistance to rotation round the node, making the problem highly unstable regarding finding an equilibrium resulting in the solution u_{x_1} "blowing up". Moreover the case of one free rotating bar with forces acting on the free node is a simple problem that can be solved without the direct stiffness method.

1.1.4:

Searched:

Axial forces $F^{(e)}$.

Solution:

The axial forces are recieved by the relations

$$F^{(e)}=k_s^{(e)}d^{(e)}=\frac{E^{(e)}A^{(e)}}{L^{(e)}}d^{(e)}=\frac{E^{(e)}A^{(e)}}{L^{(e)}}(\bar{u}_{x_j}^{(e)}-\bar{u}_{x_i}^{(e)})$$

for which we need the local displacements $\bar{\boldsymbol{u}}^{(e)}$. This is achieved with

$$\bar{\boldsymbol{u}}^{(e)}=\boldsymbol{T}^{(e)}\boldsymbol{u}^{(e)}.$$

For member (1) we get

$$\begin{bmatrix}\bar{u}_{x1}^{(1)}\\ \bar{u}_{y1}^{(1)}\\ \bar{u}_{x2}^{(1)}\\ \bar{u}_{y2}^{(1)}\end{bmatrix}=\begin{bmatrix}s & -c & 0 & 0\\ c & s & 0 & 0\\ 0 & 0 & s & -c\\ 0 & 0 & c & s\end{bmatrix}\begin{bmatrix}\frac{HL}{2EAcs^2}\\ -\frac{PL}{EA(1+2c^3)}\\ 0\\ 0\end{bmatrix}$$

$$F^{(1)} = \frac{EA}{L/c}\left(\bar{u}_{x_1}^{(1)} - \bar{u}_{x_2}^{(1)}\right) = \frac{EAc}{L}\left(\frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}\right) = \frac{H}{2s} + \frac{Pc^2}{1+2c^3}.$$

For member (2) we get

$$\begin{bmatrix} \bar{u}_{x1}^{(2)} \\ \bar{u}_{y1}^{(2)} \\ \bar{u}_{x3}^{(2)} \\ \bar{u}_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{HL}{2EAcs^2} \\ -\dfrac{PL}{EA(1+2c^3)} \\ 0 \\ 0 \end{bmatrix}$$

$$F^{(2)} = \frac{EA}{L}\left(\bar{u}_{x_1}^{(2)} - \bar{u}_{x_3}^{(2)}\right) = \frac{EAc}{L}\left(\frac{PL}{EA(1+2c^3)} - 0\right) = \frac{Pc}{1+2c^3}.$$

For member (3) we get

$$\begin{bmatrix} \bar{u}_{x1}^{(3)} \\ \bar{u}_{y1}^{(3)} \\ \bar{u}_{x2}^{(3)} \\ \bar{u}_{y2}^{(3)} \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} \dfrac{HL}{2EAcs^2} \\ -\dfrac{PL}{EA(1+2c^3)} \\ 0 \\ 0 \end{bmatrix}$$

$$F^{(3)} = \frac{EA}{L/c}\left(\bar{u}_{x_4}^{(3)} - \bar{u}_{x_1}^{(3)}\right) = \frac{EAc}{L}\left(-\frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}\right) = -\frac{H}{2s} + \frac{Pc^2}{1+2c^3}.$$

The solutions $F^{(1)}$ and $F^{(3)}$ ”blow up” for $\alpha \to 0$. This represents the situation where all bars are vertical and therefore there exists no axial forces preventing rotation around the common node. Note that the common node would be placed where node 3 is in the figure. No prevention of axial rotation would make an equilibrium highly unstable.

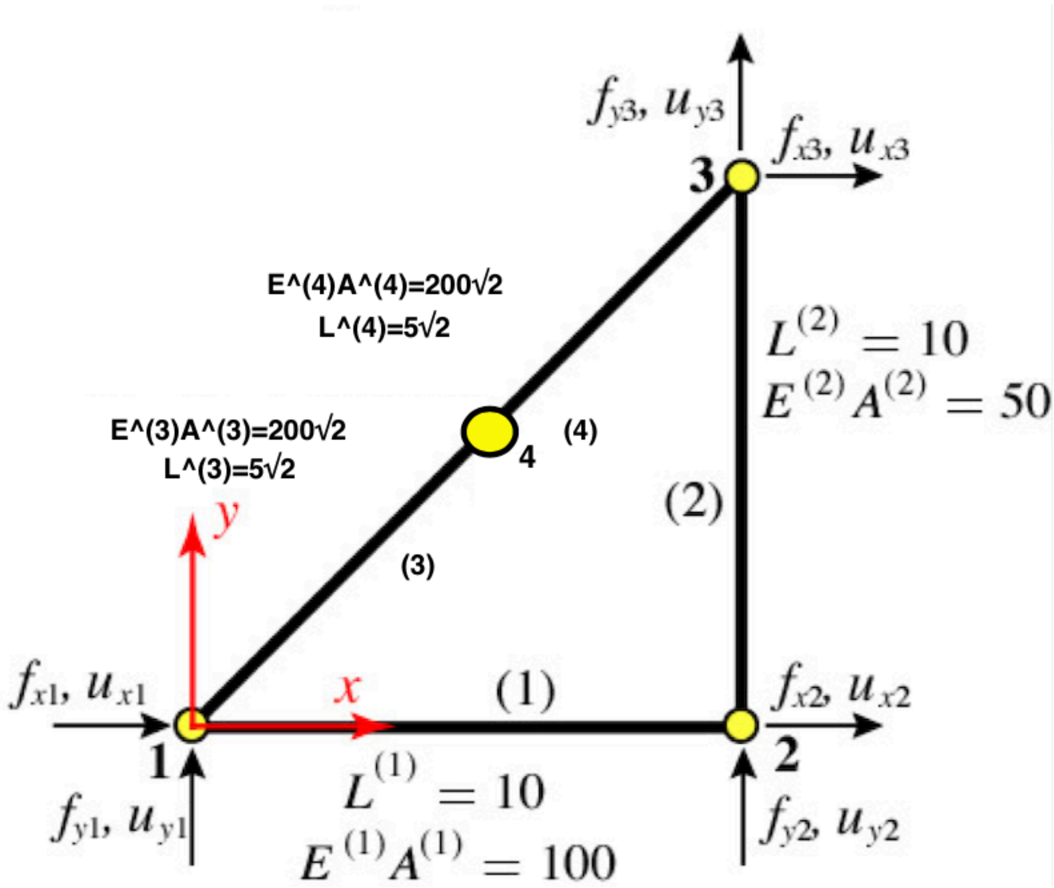
1.2:

Searched:

Try Dr. Who’s suggestion by hand computations and verify that the solution ”blows up” because the modified master stiffness is singular. Explain physically.

Solution:

The problem from the first lesson, modified with the extra node according to Dr. Who’s instructions yield the geometry in the following figure.



In the same manner as in task 1.1, using the values for $E^{(e)}, A^{(e)}$ and $L^{(e)}$ from the figure, we obtain

$$\overline{\boldsymbol{K}}^{(1)} = \boldsymbol{K}^{(1)} = 10\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\overline{\boldsymbol{K}}^{(2)} = 5\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\overline{\boldsymbol{K}}^{(3)} = 40\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\overline{\boldsymbol{K}}^{(4)} = 40\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Translating from local to global coordinates for members (2), (3) and (4) yields the translation matrices

$$\boldsymbol{T}^{(2)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\boldsymbol{T}^{(3)} = \boldsymbol{T}^{(4)} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ -0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & -0.5 & 0.5 \end{bmatrix}.$$

Using the translation matrices together with the local stiffness matrices the resulting global stiffness matrices become

$$\boldsymbol{K}^{(2)} = \left(\boldsymbol{T}^{(2)}\right)^T \overline{\boldsymbol{K}}^{(2)} \boldsymbol{T}^{(2)} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} 5 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} =$$
$$= 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{K}^{(4)} = \boldsymbol{K}^{(3)} = \left(\boldsymbol{T}^{(3)}\right)^T \overline{\boldsymbol{K}}^{(3)} \boldsymbol{T}^{(3)}$$
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} 40 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$
$$= 40 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}.$$

Expanding the stiffness matrices and using assembly as in task 1.1 the final master stiffness equations become

$$\begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & & & 25 & -20 & -20 \\ & & & & & & 40 & 40 \\ symm & & & & & & & 40 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix}.$$

The last two rows and columns are identical hence the matrix consists of dependent rows and is singular. In other words, the system cannot be solved due to lack of coundary conditions. Adding an extra node adds degrees of freedom to the system which means that without also adding extra boundary conditions the system will have too many degrees of freedom to be able to solve.