UNIVERSITAT POLITÈCNICA DE CATALUNYA

MASTER IN COMPUTATION MECHANICS AND NUMERICAL METHODS IN ENGINEERING

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

Assignment 1

by

Renan Alessio

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1- Introduction

The goal of the assignment is to apply the Direct Stiffness Method to analyze different trusses. The solution of the proposed assignments is presented and a discussion of the applied method was considered.

2 – Assingment 1

2.1 – Point A

Considering the Truss depicted in Figure 1, where L is the distance between nodes 1 and 3, it is possible to apply the Direct Stiffness Method to each bar and write the Element Stiffness Equation (1) for each bar :

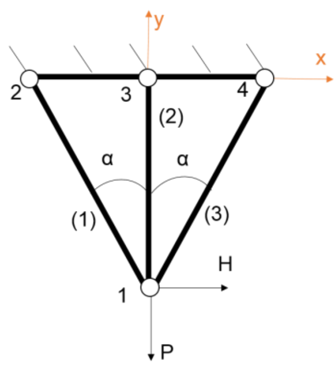

Figure 1. The truss considered for assignment 1

$$\boldsymbol{f}^{(e)} = \boldsymbol{K}^{(e)}\boldsymbol{u}^{(e)} \tag{1}$$

Equation (1) is written considering the global coordinate axis (in orange in Figure 1). The Element Stiffness Matrix $\mathbf{K}^{(e)}$ can be written in terms of the local coordinate Element Stiffness Matrix $\mathbf{K}_{local}^{(e)}$ and the Element Rotation Matrix $\mathbf{T}$. Figure 2 depicts, as an example, the local coordinate system of the element 3 Equation (2) presents such relationship:

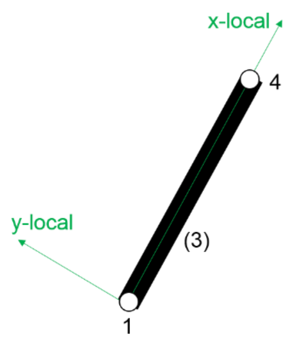

Figure 2. Local coordinate system (in green) of element 3

$$\boldsymbol{K}^{(e)} = (\boldsymbol{T}^{(e)})^{\mathrm{T}}\, \boldsymbol{K}_{local}^{(e)}\, \boldsymbol{T}^{(e)} \tag{2}$$

Where

$$\mathbf{K}_{\text{local}}^{(e)} = (EA/L)^{(e)} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{T}^{(e)} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

Equation 2 is applied to each bar element presented in Figure 1 to define their Element Stiffness Matrix $\mathbf{K}^{(e)}$:

$$\mathbf{K}^{(1)} = (EA/L) \begin{bmatrix} cs^2 & -c^2s & -cs^2 & sc^2 \\ & c^3 & sc^2 & -c^3 \\ & & cs^2 & -c^2s \\ \text{(SYMM)} & & & c^3 \end{bmatrix}$$

$$\mathbf{K}^{(2)} = (EA/L) \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ (SYMM) & & & 1 \end{bmatrix}$$

$$\mathbf{K}^{(3)} = (EA/L) \begin{bmatrix} cs^2 & c^2s & -cs^2 & -c^2s \\ & c^3 & -c^2s & -c^3 \\ & & cs^2 & c^2s \\ (SYMM) & & & c^3 \end{bmatrix}$$

Since the three Element Stiffness Matrices are defined, it is possible to assemble the global stiffness matrix **K** and the master stiffness equations **f** = **Ku**. The assembly of the master stiffness equations is achieved by applying compatibility for displacements at joints of the truss and equilibrium between internal and external forces acting on the structure. Equation 3 presents how compatibility and equilibrium are applied between the bar elements through the Equilibrium Rule to obatin the master stiffness equations.

$$\boldsymbol{f} = \boldsymbol{f}^{(1)} + \boldsymbol{f}^{(2)} + \boldsymbol{f}^{(3)} = (\boldsymbol{K}^{(1)} + \boldsymbol{K}^{(2)} + \boldsymbol{K}^{(3)})\boldsymbol{u} = \boldsymbol{K}\boldsymbol{u} \qquad (3)$$

Replacing the values of $\mathbf{K}^{(1)}$, $\mathbf{K}^{(2)}$ and $\mathbf{K}^{(3)}$ in Equation 3 and considering the equilibrium between internal and external forces applied to the structure, the master stiffness equations can be written as :

$$\begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = (EA/L) \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ (SYMM) & & & & & & & c^3 \end{bmatrix} \begin{bmatrix} ux1 \\ uy1 \\ ux2 \\ uy2 \\ ux3 \\ uy3 \\ ux4 \\ uy4 \end{bmatrix}$$

It is worth mentioning that the 5th coloumn and row contain only zeros because element 2 is aligned with the global y-axis. Element 2 is a bar element, it only has displacement along its length direction (local x-axis direction), and since its length is aligned with global y-axis, there is no dispacement in the global x-axis direction. That is why all the coefficients in the global stiffness matrix which multiply the displacement u_{x3} are zero.

2.2 – Point B

Applying the following boundary conditions :

$$u_{x2} = u_{y2} = u_{x3} = u_{y3} = u_{x4} = u_{y4} = 0$$

the master stiffness equations reduce to the following equations :

$$\frac{EA}{L}\,(2cs^2)\,u_{x1} = H \qquad (4)$$

$$\frac{EA}{L}\,(1 + 2c^3)\,u_{y1} = -P \qquad (5)$$

2.3 – Point C

To find the displacements of node 1 we solve the Equations 5 and 6 :

$$u_{x1} = \left(\frac{HL}{EA}\right)\left(\frac{1}{2cs^2}\right)$$

$$u_{y1} = \left(\frac{-PL}{EA}\right)\left(\frac{1}{2c^3 + 1}\right)$$

It is also possible to check how the solution for the displacements behave in the limit cases of α→0 and α→π/2.

<u>Considering α→0 :</u>

$$u_{x1} \rightarrow \infty$$

$$u_{y1} \rightarrow \left(\frac{-PL}{EA}\right)\left(\frac{1}{3}\right)$$

In the case of the displacement u_{x1}, its value tends to infinity due to the sinα in its denominator, which approaches the value 0 as α tends to 0, and considering H ≠ 0. Such behavior makes physical sense, because as α approaches zero, the elements 1 and 3 tend to become aligned to the global y-axis. In such scenario the structure would behave as a pendulum, there would be no resistance (no stiffness) to the force H in the global x-direction and the displacement u_{x1} would tend to infinity. Such expected behavior is due to the feature of the bar element, which only has stiffness along its length direction.

Regarding the displacement u_{y1}, the behavior in this limit case makes physical sense, because the three elements would overlap each other while being aligned with the global y-direction. In such scenario, the stiffness in the global y-direction would be three times greater (the sitffness of each element would be summed over) and so the displacement should be three times smaller (Hooke's Law).

<u>Considering α→ π/2 :</u>

$$u_{x1} \rightarrow \infty$$

$$u_{y1} \rightarrow \left(\frac{-PL}{EA}\right)$$

For the displacement u_{x1}, its value also tends to infinity due to the cosα in its denominator, which approaches the value 0 as α tends to π/2, considering H ≠ 0. Such behavior also makes physical sense, since in this limit case the elements 1 and 3 are parallel with the global x-axis and the legth of both elements would tend to infinity. In such scenario, the stiffness of both elements would tend to zero and there would be no resistance to the force H and the displacement u_{x1} would tend to infinity.Again, such expected behavior is due to the feature of the bar element, which only has stiffness along its length direction. For the displacement u_{y1}, the behavior presented in the limit case makes physical sense, since the element 2 is the only element aligned with the global y-axis. In such scenario, only the stiffness of element 2 would be considered to calculate the displacement u_{y1}.

2.4 – Point D

To recover the axial forces it is necessary to calculate the displacements according to the local axis of each element with the following relationship :

$$\boldsymbol{u}_{\boldsymbol{local}}^{(\boldsymbol{e})} = \boldsymbol{T}^{(\boldsymbol{e})}\boldsymbol{u}^{(\boldsymbol{e})} \quad (6)$$

Since the bar elements have internal force only along the local x-axis (direction of stiffness), the axial (internal) forces can be calculated with the following relationship :

$$\boldsymbol{F}^{(\boldsymbol{e})} = \frac{\boldsymbol{E}^{(\boldsymbol{e})}\boldsymbol{A}^{(\boldsymbol{e})}}{\boldsymbol{L}^{(\boldsymbol{e})}}\boldsymbol{d}^{(\boldsymbol{e})} \quad (7)$$

Where $d^{(e)}$ is the difference between the axial (local x-axis) displacements of the element's nodes :

$$\boldsymbol{d^{(e)} = u^{(e)}_{local/xj} - u^{(e)}_{local/xi}} \tag{8}$$

Applying Equations 6-8, the internal force of each element can be calculated :

<u>For element 1 :</u>

$$\boldsymbol{u^{(1)}_{local/x1} = -\frac{L}{EA}\left(\frac{H}{2cs} + \frac{Pc}{2c^3 + 1}\right)}$$

$$\boldsymbol{u^{(1)}_{local/x2} = 0}$$

$$\boldsymbol{d^{(1)} = u^{(1)}_{local/x2} - u^{(1)}_{local/x1} = \frac{L}{EA}\left(\frac{H}{2cs} + \frac{Pc}{2c^3 + 1}\right)}$$

$$\boldsymbol{F^{(1)} = \left(\frac{H}{2s} + \frac{Pc^2}{2c^3 + 1}\right)}$$

<u>For element 2 :</u>

$$\boldsymbol{u^{(2)}_{local/x1} = -\frac{L}{EA}\left(\frac{P}{2c^3 + 1}\right)}$$

$$\boldsymbol{u^{(2)}_{local/x3} = 0}$$

$$\boldsymbol{d^{(2)} = u^{(2)}_{local/x3} - u^{(2)}_{local/x1} = \frac{L}{EA}\left(\frac{P}{2c^3 + 1}\right)}$$

$$\boldsymbol{F^{(2)} = \left(\frac{P}{2c^3 + 1}\right)}$$

<u>For element 3 :</u>

$$\boldsymbol{u^{(3)}_{local/x1} = \frac{L}{EA}\left(\frac{H}{2cs} - \frac{Pc}{2c^3 + 1}\right)}$$

$$u^{(3)}_{local/x4} = 0$$

$$d^{(3)} = u^{(3)}_{local/x4} - u^{(3)}_{local/x1} = -\frac{L}{EA}\left(\frac{H}{2cs} - \frac{Pc}{2c^3 + 1}\right)$$

$$F^{(3)} = \left(-\frac{H}{2s} + \frac{Pc^2}{2c^3 + 1}\right)$$

It is important to state that $F^{(1)}$ and $F^{(3)}$ tend to infinity when $H \neq 0$ and $\alpha \rightarrow 0$, because in such limit case the displacement in the global x-direction of node 1 tends to infinity as well (elements 1 and 3 would not offer resistance to the external load H).

3 - Assignment 2

Considering the new truss depicted in Figure 3 it is possbile to obtain the element stiffness matrices :

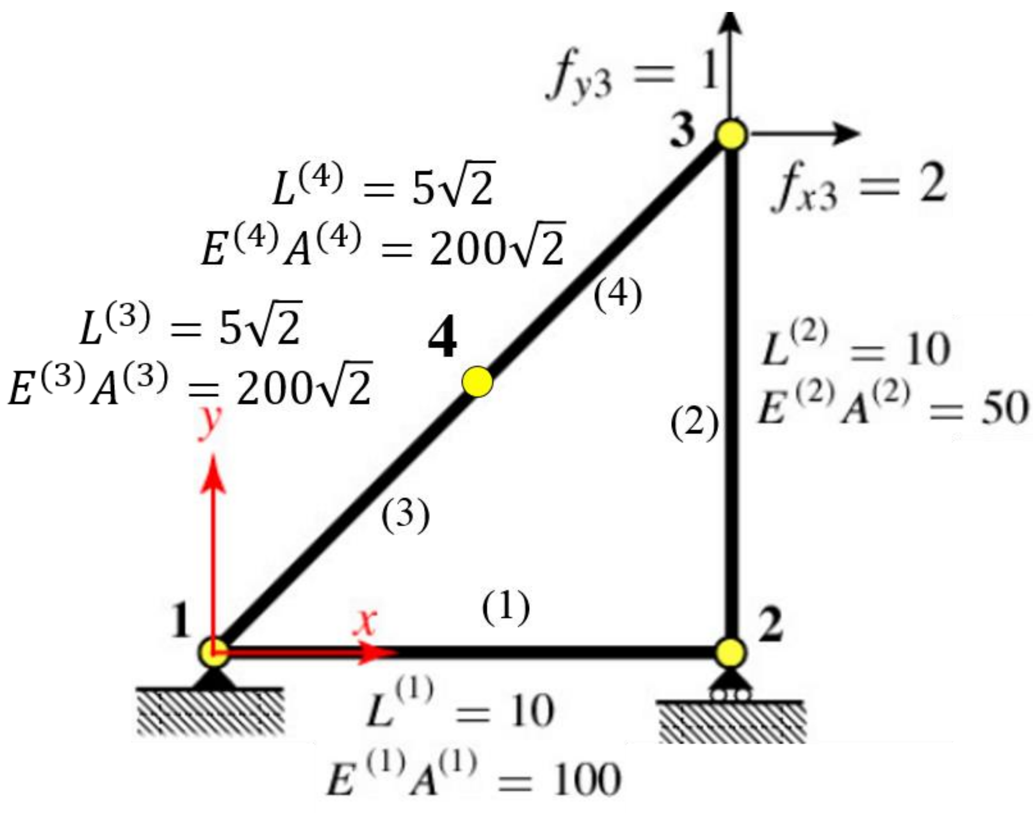

Figure 3. Truss considered for assignment 2.

$$\mathbf{K}^{(1)} = \begin{bmatrix} 10 & 0 & -10 & 0 \\ & 0 & 0 & 0 \\ & & 10 & 0 \\ \text{(SYMM)} & & & 0 \end{bmatrix}$$

$$\mathbf{K}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 5 & 0 & -5 \\ & & 0 & 0 \\ \text{(SYMM)} & & & 5 \end{bmatrix}$$

$$\mathbf{K}^{(3)} = \begin{bmatrix} 20 & 20 & -20 & -20 \\ & 20 & -20 & -20 \\ & & 20 & 20 \\ \text{(SYMM)} & & & 20 \end{bmatrix}$$

$$\mathbf{K}^{(4)} = \begin{bmatrix} 20 & 20 & -20 & -20 \\ & 20 & -20 & -20 \\ & & 20 & 20 \\ \text{(SYMM)} & & & 20 \end{bmatrix}$$

Considering the element stiffness matrices and applying Equations 1-3 as well as compatibility for nodal displacement and equilibrium between internal and external forces, the following master stiffness equations are obtained :

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & -10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & & & 25 & -20 & -20 \\ \text{(SYMM)} & & & & & & 40 & 20 \\ & & & & & & & 20 \end{bmatrix} \begin{bmatrix} ux1 \\ uy1 \\ ux2 \\ uy2 \\ ux3 \\ uy3 \\ ux4 \\ uy4 \end{bmatrix}$$

Applying the following boundary conditions to the master stiffeness equations

$$u_{x1} = u_{y1} = u_{y2} = 0$$

the resulting system of equations is :

$$
\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & -20 & -20 \\ 0 & 20 & 25 & -20 & -20 \\ 0 & 20 & -20 & 40 & 40 \\ 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} ux2 \\ ux3 \\ uy3 \\ ux4 \\ uy4 \end{bmatrix}
$$

The resulting system of equations is singular since there are linear dependent columns and, therefore, cannot be solved. The linear dependent columns are the 4th and 5th, which are related to the degrees of freedom of node 4 (the extra node). At such node, there is only one force acting on it (internal force of element 3 before node 4 was added) and there are no prescribed displacement restrictions, therefore the equilibrium rule used in the Direct Stiffness Method is not met at that node. Such behavior makes the solution "blow up".

4 – Discussion and Extensions

The Direct Stiffness Method has its formulation based on the equilibrium rule between the elements. The method presented itself to be very uselful when applied to structures built with 1D bars in 1 or 2D space. Accounting for the 1D bars, the equilibrium equations can be obtained in a simple manner and the compatibility between the element can be enforced naturally. It is important to mention that, since the element is a 1D bar, the constitutive behavior between stress and strain reduces to an equivalent spring stiffness (Hooke's Law). Therefore, structures built with trusses can be analyzed in a simple manner by applying the Direct Stiffness Method. Nevertheless, when considering 2D elements, the equilibrium equations would be much more complicated to be obtained and the Direct Stiffness Method becomes impracticable. In such cases, other approaches should applied in order to obtain the displacement field.

Appendix 1 – Solution of Classwork

Considering the structure depicted in Figure 1A and L = 6m, A = 6cm², E=200GPa F = 80KN :

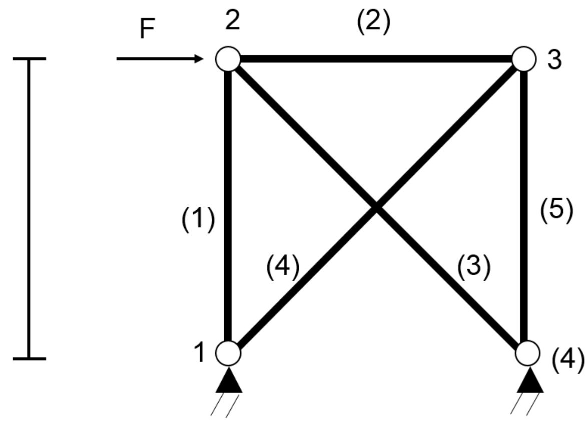

Figure 1A. Structure under analysis.

It is possible to calculate each element stiffness matrix :

$$\boldsymbol{K}^{(1)} = 10^6 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 20 & 0 & -20 \\ 0 & 0 & 0 & 0 \\ 0 & -20 & 0 & 20 \end{pmatrix}$$

$$\boldsymbol{K}^{(2)} = 10^6 \begin{pmatrix} 20 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 \\ -20 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{K}^{(3)} = 10^6 \begin{pmatrix} 7.071 & -7.071 & -7.071 & 7.0710 \\ -7.071 & 7.071 & 7.071 & -7.071 \\ -7.071 & 7.071 & 7.071 & -7.071 \\ 7.071 & -7.071 & -7.071 & 7.071 \end{pmatrix}$$

$$\boldsymbol{K}^{(4)} = 10^6 \begin{pmatrix} 7.071 & 7.071 & -7.071 & -7.0710 \\ 7.071 & 7.071 & -7.071 & -7.071 \\ -7.071 & -7.071 & 7.071 & 7.071 \\ -7.071 & -7.071 & 7.071 & 7.071 \end{pmatrix}$$

$$\boldsymbol{K}^{(5)} = 10^6 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 20 & 0 & -20 \\ 0 & 0 & 0 & 0 \\ 0 & -20 & 0 & 20 \end{pmatrix}$$

Applying the equilibium between external and internal forces and imposing the compatibility at nodes's diplacements, the master stiffness equations can be writeen as :

$$\begin{bmatrix} 0 \\ 0 \\ 80000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 10^{\wedge}(6) \begin{bmatrix} 7.071 & 7.071 & 0 & 0 & -7.071 & -7.071 & 0 & 0 \\ & 27.071 & 0 & -20 & -7.071 & -7.071 & 0 & 0 \\ & & 27.071 & -7.071 & -20 & 0 & -7.071 & 7.071 \\ & & & 27.071 & 0 & 0 & 7.071 & -7.071 \\ & & & & 27.071 & 7.071 & 0 & 0 \\ & & & & & 27.071 & 0 & -20 \\ & \text{(SYMM)} & & & & & 7.071 & -7.071 \\ & & & & & & & 27.071 \end{bmatrix} \begin{bmatrix} ux1 \\ uy1 \\ ux2 \\ uy2 \\ ux3 \\ uy3 \\ ux4 \\ uy4 \end{bmatrix}$$

Applying the following boundary conditions to the master stiffness equations, the reduced stiffness equations are obtained :

$$u_{x1} = u_{y1} = u_{x4} = u_{y4} = 0$$

$$\begin{bmatrix} 80000 \\ 0 \\ 0 \\ 0 \end{bmatrix} = (10^{\wedge}6) \begin{bmatrix} 27.071 & -7.071 & -20 & 0 \\ -7.071 & 27.071 & 0 & 0 \\ -20 & 0 & 27.071 & 7.071 \\ 0 & 0 & 7.071 & 27.071 \end{bmatrix} \begin{bmatrix} ux2 \\ uy2 \\ ux3 \\ uy3 \end{bmatrix}$$

Solving for the displacement u_{x2}, $u_{y2,}$ u_{x3} and u_{y3} and writting the full displacement vector:

$$\underline{u}^T = 10^{-3\,*} [\ 0 \quad 0 \quad 8.54 \quad 2.23 \quad 6.77 \quad -1.77 \quad 0 \quad 0\ ]^T \text{ [m]}$$