HOMEWORK 5

Assignment 5.1: Solid elements

1) Compute f^e.

$$f^e_{ext} = \int_{Ve} N^T\, b\, dV = \int_{Ve} \begin{bmatrix} \xi_1 & 0 & 0 & \xi_2 & 0 & 0 & \xi_3 & 0 & 0 & \xi_4 & 0 & 0 \\ 0 & \xi_1 & 0 & 0 & \xi_2 & 0 & 0 & \xi_3 & 0 & 0 & \xi_4 & 0 \\ 0 & 0 & \xi_1 & 0 & 0 & \xi_2 & 0 & 0 & \xi_3 & 0 & 0 & \xi_4 \end{bmatrix}^T b\, dV =$$

$$N = \begin{bmatrix} \xi_1 & 0 & 0 & \xi_2 & 0 & 0 & \xi_3 & 0 & 0 & \xi_4 & 0 & 0 \\ 0 & \xi_1 & 0 & 0 & \xi_2 & 0 & 0 & \xi_3 & 0 & 0 & \xi_4 & 0 \\ 0 & 0 & \xi_1 & 0 & 0 & \xi_2 & 0 & 0 & \xi_3 & 0 & 0 & \xi_4 \end{bmatrix}$$

$$b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} 0 \\ b_y \\ 0 \end{bmatrix}$$

$$\int_V \xi\, dV = \frac{V}{4}$$

$$= \frac{V}{4} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ b_y \\ 0 \end{bmatrix}$$

<u>**Assignment 5.2:**</u> FEM Convergence requirements

$$\mathrm{e} = \mathrm{B\,u}$$

$$B = \frac{1}{J}\begin{bmatrix} \frac{dN_1}{d\xi} \\ \frac{dN_2}{d\xi} \\ \frac{dN_3}{d\xi} \end{bmatrix} = \frac{1}{L\left(\frac{1}{2} - 2\alpha\xi\right)}\begin{bmatrix} \frac{2\xi - 1}{2} \\ \frac{2\xi + 1}{2} \\ -2\xi \end{bmatrix}$$

1) **Show that the minimum α for which J vanishes at a point in the element are ±¼.**

$$\mathrm{J} = L\left(\frac{1}{2} - 2\alpha\xi\right) = 0 \rightarrow \frac{1}{2} - 2\alpha\xi = 0 \rightarrow \alpha = \frac{1}{4\xi}$$

$$\xi \in [-1,1]$$

$$\alpha \in \left[-\frac{1}{4}, \frac{1}{4}\right]$$

2) **Interpret this result as singularity by showing that the axial strain becomes infinite at an end point.**

The values of ξ are -1 and 1 at the end of the element.

- ξ = 1

$$\lim_{\alpha\to\frac{1}{4}} \frac{1}{J} = \lim_{\alpha\to\frac{1}{4}} \frac{1}{L\left(\frac{1}{2} - 2\alpha\right)} = \frac{1}{0}$$

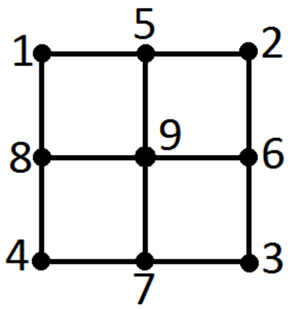
$$\lim_{\alpha\to-\frac{1}{4}} \frac{1}{J} = \lim_{\alpha\to-\frac{1}{4}} \frac{1}{L\left(\frac{1}{2} - 2\alpha\right)} = \frac{1}{L}$$

- ξ = -1

$$\lim_{\alpha\to\frac{1}{4}} \frac{1}{J} = \lim_{\alpha\to\frac{1}{4}} \frac{1}{L\left(\frac{1}{2} + 2\alpha\right)} = \frac{1}{L}$$

$$\lim_{\alpha\to-\frac{1}{4}} \frac{1}{J} = \lim_{\alpha\to-\frac{1}{4}} \frac{1}{L\left(\frac{1}{2} + 2\alpha\right)} = \frac{1}{0}$$

Assignment 5.3: FEM Convergence requirements



$x_1 = x_4 = x_8 = -L$ $\quad$ $y_1 = y_5 = y_2 = L$

$x_9 = x_7 = 0$ $\quad$ $y_8 = y_9 = y_6 = 0$

$x_2 = x_6 = x_3 = L$ $\quad$ $y_4 = y_7 = y_3 = -L$

$x_5 = \alpha L$

- Corner nodes: i = 1,2,3,4.

$$\frac{dN}{d\xi} = \frac{1}{4}(\xi - 1)(\eta - 1)\eta + \frac{1}{4}(\eta - 1)\xi\eta$$
$$\frac{dN}{d\eta} = \frac{1}{4}(\xi - 1)(\eta - 1)\xi + \frac{1}{4}(\xi - 1)\xi\eta$$

- Central node: i = 9.

$$\frac{dN}{d\xi} = 2\xi(\eta^2 - 1)$$
$$\frac{dN}{d\xi} = 2\eta(\xi^2 - 1)$$

- Mid-sides node: i = 5,6,7,8.

$$\frac{dN}{d\xi} = \frac{1}{2}(1 - \eta^2)(2\xi - 1)$$

$$\frac{dN}{d\eta} = \xi\eta(1 - \xi)$$

The Jacobian matrix is defined as,

$$J = \begin{bmatrix} \frac{dx}{d\xi} & \frac{dx}{d\eta} \\ \frac{dy}{d\xi} & \frac{dy}{d\eta} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1 - \eta^2)(2\xi - 1) & \xi\eta(1 - \xi) \\ 0 & 0 \end{bmatrix} \alpha \frac{L}{2} = 0$$

$$\frac{dx}{d\xi} = \frac{dN}{d\xi}(x_1 + x_2 + x_3 + x_4) + \frac{dN}{d\xi}x_9 + \frac{dN}{d\xi}(x_5 + x_6 + x_7 + x_8) = \frac{1}{2}(1 - \eta^2)(2\xi - 1)\,\alpha\frac{L}{2}$$

$$\frac{dx}{d\eta} = \frac{dN}{d\eta}(x_1 + x_2 + x_3 + x_4) + \frac{dN}{d\eta}x_9 + \frac{dN}{d\eta}(x_5 + x_6 + x_7 + x_8) = \xi\eta(1 - \xi)\alpha\frac{L}{2}$$

$$\frac{dx}{d\xi} = \frac{dN}{d\xi}(y_1 + y_2 + y_3 + y_4) + \frac{dN}{d\xi}y_9 + \frac{dN}{d\xi}(y_5 + y_6 + y_7 + y_8) = 0$$

$$\frac{dx}{d\eta} = \frac{dN}{d\eta}(y_1 + y_2 + y_3 + y_4) + \frac{dN}{d\eta}y_9 + \frac{dN}{d\eta}(y_5 + y_6 + y_7 + y_8) = 0$$

The determinant of the Jacobian is always zero for any α.