HOMEWORK 3

Assignment 3.1: The plane Stress Problem

1) Find the inverse relations for E, ν in terms of μ, λ.

E(μ, λ), ν(μ, λ) are calculated solving the following system:

$$\begin{cases} E = 2\mu(1+\nu) \\ E = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} \end{cases}$$

The first step will be calculating ν(μ, λ),

$$2\mu(1+\nu) = \frac{\lambda(1+\nu)(1-2\nu)}{\nu}$$

$$2\mu\nu = \lambda(1-2\nu)$$

$$\nu(2\mu+2\lambda) = \lambda \longrightarrow \boxed{\nu = \frac{\lambda}{2(\mu+\lambda)}}$$

Substituting ν into any of the previous equation,

$$E = 2\mu\left(1+\frac{\lambda}{2(\mu+\lambda)}\right) = \frac{2\mu+2\lambda+\lambda}{2(\mu+\lambda)}2\mu = \boxed{\mu\frac{2\mu+3\lambda}{\mu+\lambda}}$$

2) Express the elastic matrix for plane stress and plane strain cases in terms of μ, λ.

A. PLAIN STRESS:

The constitutive equation for isotropic material is,

$$[\sigma] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}[\varepsilon]$$

Substituting E and ν for the expressions calculated in the previous part,

- $$\frac{E}{1-\nu^2} = \frac{\mu(2\mu+3\lambda)}{(\mu+\lambda)\left[1-\left(\frac{\lambda}{2(\mu+\lambda)}\right)^2\right]} = \frac{\mu(2\mu+3\lambda)}{(\mu+\lambda)\left[\frac{4(\mu+\lambda)^2-\lambda^2}{4(\mu+\lambda)^2}\right]} = \frac{\mu(2\mu+3\lambda)4(\mu+\lambda)}{4(\mu+\lambda)^2-\lambda^2} = \frac{\mu(2\mu+3\lambda)4(\mu+\lambda)}{4\mu^2+8\mu\lambda+4\lambda^2-\lambda^2} = \frac{\mu 4(\mu+\lambda)}{\frac{4\mu^2+8\mu\lambda+3\lambda^2}{(2\mu+3\lambda)}} = \frac{4\mu(\mu+\lambda)}{2\mu+\lambda}$$

- $$\frac{E\nu}{1-\nu^2} = \frac{4\mu(\mu+\lambda)}{2\mu+\lambda}\frac{\lambda}{2(\mu+\lambda)} = \frac{2\mu\lambda}{2\mu+\lambda}$$

- $$\frac{\mathrm{E}}{1-\nu^2}\frac{1-\nu}{2}=\frac{4\mu(\mu+\lambda)}{2\mu+\lambda}\frac{1-\frac{\lambda}{2(\mu+\lambda)}}{2}=\frac{2\mu(\mu+\lambda)\left[\frac{2(\mu+\lambda)-\lambda}{2\mu+\lambda}\right]}{2\mu+\lambda}=\frac{\mu(2\mu+\lambda)}{2\mu+\lambda}=\mu$$

The elastic matrix for plain stress cases in terms of μ, λ will be defined as,

$$[\mathrm{E}]=\frac{\mu}{2\mu+\lambda}\begin{bmatrix}4(\mu+\lambda) & 2\lambda & 0\\ 2\lambda & 4(\mu+\lambda) & 0\\ 0 & 0 & 2\mu+\lambda\end{bmatrix}$$

B. PLANE STRAIN:

The constitutive equation for plain strain cases is,

$$[\sigma]=\frac{\mathrm{E}(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix}1 & \frac{\nu}{1-\nu} & 0\\ \frac{\nu}{1-\nu} & 1 & 0\\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)}\end{bmatrix}[\varepsilon]$$

The members of the new matrix depending on μ, λ are,

- $$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}=\frac{\mu(2\mu+3\lambda)\left(1-\frac{\lambda}{2(\mu+\lambda)}\right)}{(\mu+\lambda)\left(1+\frac{\lambda}{2(\mu+\lambda)}\right)\left(1-\frac{\lambda}{\mu+\lambda}\right)}=\frac{\mu(2\mu+3\lambda)\left(\frac{2\mu+2\lambda-\lambda}{2(\mu+\lambda)}\right)}{\frac{2\mu+3\lambda}{2(\mu+\lambda)}(\mu+\lambda-\lambda)}=$$

$$=\frac{\mu(2\mu+3\lambda)2(\mu+\lambda)\left(\frac{2\mu+\lambda}{2(\mu+\lambda)}\right)}{\mu(2\mu+3\lambda)}=2\mu+\lambda$$

- $$\frac{E(1-\nu)\nu}{(1+\nu)(1-2\nu)(1-\nu)}=\frac{\mu(2\mu+3\lambda)\lambda}{(\mu+\lambda)2(\mu+\lambda)\left(1+\frac{\lambda}{2(\mu+\lambda)}\right)\left(1-\frac{\lambda}{\mu+\lambda}\right)}=$$

$$=\frac{\mu\lambda(2\mu+3\lambda)}{2\left(\mu+\lambda+\frac{\lambda}{2}\right)(\mu+\lambda-\lambda)}=\frac{\mu\lambda(2\mu+3\lambda)}{2\mu\left(\mu+\frac{3\lambda}{2}\right)}=\lambda$$

- $$\frac{E(1-\nu)(1-2\nu)}{(1+\nu)(1-2\nu)(1-\nu)}=\frac{\mu(2\mu+3\lambda)}{(\mu+\lambda)\left(1+\frac{\lambda}{2(\mu+\lambda)}\right)2}=\frac{\mu(2\mu+3\lambda)}{\left(\mu+\lambda+\frac{\lambda}{2}\right)2}=$$

$$=\frac{\mu(2\mu+3\lambda)}{2\mu+3\lambda}=\mu$$

Substituting these new values in the elastic matrix,

$$[\mathrm{E}]=\begin{bmatrix}2\mu+\lambda & \lambda & 0\\ \lambda & 2\mu+\lambda & 0\\ 0 & 0 & \mu\end{bmatrix}$$

3) Split the stress-strain matrix E for plane strain as $E_\lambda + E_\mu$.

$$\mathrm{E} = \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} = E_\lambda + E_\mu = \boxed{\begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}}$$

4) Express E_λ and E_μ also in terms of E and v.

$$E_\lambda = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boxed{\frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

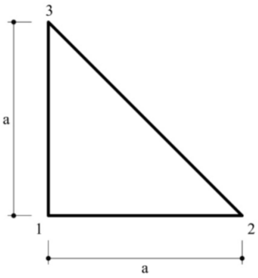
$$E_\mu = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

<u>Assignment 3.2:</u> The 3-node Plane Stress Triangle

1) Calculate the stiffness matrixes K_{tri} and K_{bar} for both discrete models.

- **A PLANE LINEAR TRIANGLE:**

	1	2	3
x	0	1	0
y	0	0	1



The stiffness matrix for constant thickness plane is given by the formula,

$$\mathbf{K}^e = A\,h\,\mathbf{B}^T\,\mathbf{E}\,\mathbf{B} = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}.$$

$$A = \frac{a^2}{2}$$

$$x_{jk} = x_j - x_k$$

$$y_{jk} = y_j - y_k$$

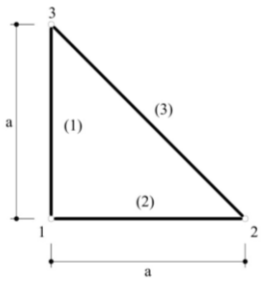
$$\text{Plane stress case: } E(\nu = 0) = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E/2 \end{bmatrix}$$

Replacing the values in the matrix,

$$K = \frac{1}{2}\begin{bmatrix} 0-1 & 0 & 0-1 \\ 0 & 0-1 & 0-1 \\ 1-0 & 0 & 0 \\ 0 & 0 & 1-0 \\ 0 & 0 & 1-0 \\ 0 & 1-0 & 0 \end{bmatrix} \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E/2 \end{bmatrix} \begin{bmatrix} 0-1 & 0 & 1-0 & 0 & 0 & 0 \\ 0 & 0-1 & 0 & 0 & 0 & 1-0 \\ 0-1 & 0-1 & 0 & 1-0 & 1-0 & 0 \end{bmatrix}$$

$$= E\begin{bmatrix} 3/4 & 1/4 & -1/2 & -1/4 & -1/4 & 0 \\ 1/4 & 3/4 & 0 & -1/4 & -1/4 & -1/2 \\ -1/2 & 0 & 1/2 & 0 & 0 & 0 \\ -1/4 & -1/4 & 0 & 1/4 & 1/4 & 0 \\ -1/4 & -1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & -1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$

- **BAR ELEMENTS:** $A_1 = A_2 \neq A_3$



The stiffness matrix for each element is defined as,

$$K^{(e)} = \frac{AE^{(e)}}{L}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

- <u>Bar 1:</u> 1→3

$$K^{(1)} = \frac{A_1 E}{a}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

- <u>Bar 2:</u> 1→2

$$K^{(2)} = \frac{A_2 E}{a}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- <u>Bar 3:</u> 2→3

$$K^{(3)} = \frac{A_3 E}{\sqrt{2a^2}}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

The global stiffness equation is obtained substituting the data and assembling the previous matrixes,

$$K = \begin{bmatrix} k_{11}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & k_{13}^{(3)} \\ & k_{22}^{(2)} + k_{22}^{(3)} & k_{23}^{(3)} \\ Symm & & k_{33}^{(1)} + k_{33}^{(3)} \end{bmatrix}$$

$$K = E\begin{bmatrix} A_1 & 0 & -A_1 & 0 & 0 & 0 \\ & A_1 & 0 & 0 & 0 & -A_1 \\ & & A_1 + 0.35A_3 & -0.35A_3 & -0.35A_3 & 0.35A_3 \\ & & & 0.35A_3 & 0.35A_3 & -0.35A_3 \\ & & & & 0.35A_3 & -0.35A_3 \\ & & & & & 0.35A_3 + A_1 \end{bmatrix}$$

2) Is there any set of values for the cross sections A1 = A2 and A3 to make both stiffness matrix equivalent? If not, which are the values that make them more similar?

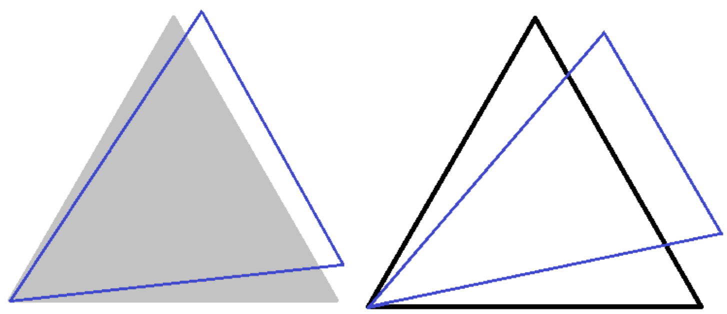
Analyzing both matrixes no direct links could be established between them and it is impossible to purpose any value of A_1 and A_3 that make them more similar.

$$K_{tri} = E\begin{bmatrix} 3/4 & 1/4 & -1/2 & -1/4 & -1/4 & 0 \\ 1/4 & 3/4 & 0 & -1/4 & -1/4 & -1/2 \\ -1/2 & 0 & 1/2 & 0 & 0 & 0 \\ -1/4 & -1/4 & 0 & 1/4 & 1/4 & 0 \\ -1/4 & -1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & -1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$K_{bar} = E\begin{bmatrix} A_1 & 0 & -A_1 & 0 & 0 & 0 \\ & A_1 & 0 & 0 & 0 & -A_1 \\ & & A_1 + 0.35A_3 & -0.35A_3 & -0.35A_3 & 0.35A_3 \\ & & & 0.35A_3 & 0.35A_3 & -0.35A_3 \\ & & & & 0.35A_3 & -0.35A_3 \\ & & & & & 0.35A_3 + A_1 \end{bmatrix}$$

3) Why these two stiffness matrixes are not equal? Find a physical explanation.

The stiffness matrix is different because one element is solid and the other one is hollow. They might have similar response when the forces acting upon the element are simple. However, when the forces are more complex, especially when they create bending or turning in the node, the answer is quite different. The solid element imposes greater resistance to the turning on the nodes than the bar element.



4) Consider nowidering v≠0 and extract some conclusions.

The poisson´s coefficient determined the strains in the different directions of the applied stress. The main consequence is that the strain applied in a certain direction will affect to the strains of the other directions. The strain suffered in the direction of the stress would be reduced and in the other direction would be greater.

$$K = \frac{1}{2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} =$$

$$= \frac{E}{2(1-\nu^2)}\begin{bmatrix} 1+\nu+\frac{1-\nu}{2} & \nu+\frac{1-\nu}{2} & -1 & -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & -\nu \\ \nu+\frac{1-\nu}{2} & 1+\frac{1-\nu}{2} & -\nu & -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & -1 \\ -1-\nu & -\nu & 1 & 0 & 0 & 1 \\ -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & \nu \end{bmatrix}$$