Assignment 4

[Structure of Revolution]

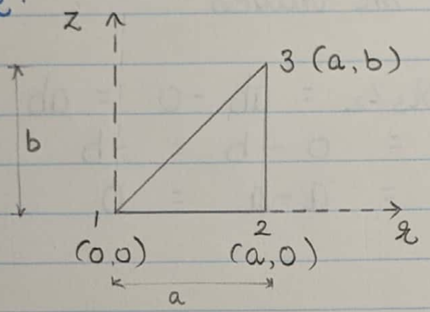
→ Assignment 4.1 [Axisymmetric triangle].

Q I) Compute the entries of k^e

→ Solution:

Step ① Element type:

Pt 1 $(r_1, z_1) = (0, 0)$
Pt 2 $(r_2, z_2) = (a, 0)$
Pt 3 $(r_3, z_3) = (a, b)$



{ 2 DOF per node
3 node - 6 DOF }

Step ② Displacement function:

$$\begin{Bmatrix} U(r,z) \\ w(r,z) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} U_1 \\ w_1 \\ U_2 \\ w_2 \\ U_3 \\ w_3 \end{bmatrix}$$

where,

$$N_1 = \frac{1}{2A}(\alpha_1 + \beta_1 r + \gamma_1 z)$$

$$N_2 = \frac{1}{2A}(\alpha_2 + \beta_2 r + \gamma_2 z)$$

$$N_3 = \frac{1}{2A}(\alpha_3 + \beta_3 r + \gamma_3 z)$$

Step ③ strain

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ 2A\frac{N_1}{r} & 0 & 2A\frac{N_2}{r} & 0 & 2A\frac{N_3}{r} & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix} \begin{bmatrix} U_1 \\ w_1 \\ U_2 \\ w_2 \\ U_3 \\ w_3 \end{bmatrix}$$

Taking derivative of shape function & substituting the values,

$$[\varepsilon] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \frac{\alpha_1+\beta_1 r+\gamma_1 z}{r} & 0 & \frac{\alpha_2+\beta_2 r+\gamma_2 z}{r} & 0 & \frac{\alpha_3+\beta_3 r+\gamma_3 z}{r} & 0 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}\begin{bmatrix} U_1 \\ \omega_1 \\ U_2 \\ \omega_2 \\ U_3 \\ \omega_3 \end{bmatrix}$$

Now calculating the values -

At Node 1

$\alpha_1 = r_2 z_3 - r_3 z_2 = ab - 0 = ab$

$\beta_1 = z_2 - z_3 = 0 - b = -b$

$\gamma_1 = r_3 - r_2 = a - a = 0$

Node 2:

$\alpha_2 = r_3 \times z_1 - r_1 z_3 = 0 - 0 = 0$

$\beta_2 = z_3 - z_1 = b - 0 = b$

$\gamma_2 = r_1 - r_3 = 0 - a = -a$

Node 3:

$\alpha_3 = r_1 z_2 - r_2 z_1 = 0 - 0 = 0$

$\beta_3 = z_1 - z_2 = 0 - 0 = 0$

$\gamma_3 = r_2 - r_1 = a - 0 = a$

Area:

$$2A = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times a \times b = \frac{ab}{2}$$

or.

$$2A = \det\begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \det\begin{vmatrix} 1 & 1 & 1 \\ 0 & a & a \\ 0 & 0 & b \end{vmatrix}$$

ie, $A = \frac{ab}{2}$

Step ④ Stiffness Matrix -

$$[K] = 2\pi \iint_A [\bar{B}]^T [\bar{D}][\bar{B}] \, r \, dr \, dz.$$

To evaluate [B] by centroid point $(\bar{r}, \bar{z})$ of the element.

ie. $[B(\bar{r}, \bar{z})] = [\bar{B}]$

& Centroid formula -

$$r = \bar{r} = \frac{r_1 + r_2 + r_3}{3} = \frac{0 + a + a}{3} = \frac{2a}{3}$$

$$z = \bar{z} = \frac{z_1 + z_2 + z_3}{3} = \frac{0 + 0 + b}{3} = \frac{b}{3}$$

& Stiffness matrix (Consideration)

The triangular subdivisions are consistent with the final stress distribution, then the stiffness matrix is

$$\boxed{[K] = 2\pi \bar{r} A [\bar{B}]^T [\bar{D}][\bar{B}]}$$

Calculating the
step ⑤ Values & substitution. -

$$\frac{\alpha_1 + \beta_1 \bar{r} + \gamma_1 \bar{z}}{\bar{r}} = \frac{3}{2a}\left(ab + (-b)\frac{2a}{3} + 0\right) = \frac{b}{2}$$

$$\frac{\alpha_2 + \beta_2 \bar{r} + \gamma_2 \bar{z}}{\bar{r}} = \frac{3}{2a}\left(0 + \frac{2ab}{3} + \frac{-ab}{3}\right) = \frac{b}{2}$$

$$\frac{\alpha_3 + \beta_3 \bar{r} + \gamma_3 \bar{z}}{\bar{r}} = \frac{3}{2a}\left(0 + 0 + \frac{ab}{3}\right) = \frac{b}{2}$$

B matrix -

$$[B] = \frac{1}{2A} \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{b}{2} & 0 & \frac{b}{2} & 0 & \frac{b}{2} & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

$$\begin{array}{ccc} B_1 & B_2 & B_3 \end{array}$$

$$\cancel{[B]^T[A][B]} = $$

$$B^T E B = \frac{E}{4A^2}\begin{bmatrix} -b & 0 & b/2 & 0 \\ 0 & 0 & 0 & -b \\ b & 0 & b/2 & -a \\ 0 & -a & 0 & b \\ 0 & 0 & b/2 & a \\ 0 & a & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ b/2 & 0 & b/2 & 0 & b/2 & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

$$B^T E B = \frac{E}{(ab)^2}\begin{bmatrix} \frac{5b^2}{4} & 0 & \frac{-3b^2}{4} & 0 & \frac{b^2}{4} & 0 \\ 0 & \frac{b^2}{2} & \frac{ab}{2} & \frac{-b^2}{2} & -\frac{ab}{2} & 0 \\ \frac{-3b^2}{4} & \frac{ab}{2} & \frac{5b^2}{4}+\frac{a^2}{2} & \frac{-ab}{2} & \frac{b^2}{4}-\frac{a^2}{2} & 0 \\ 0 & \frac{-b^2}{2} & \frac{-ab}{2} & \frac{a^2+b^2}{2} & \frac{ab}{2} & -a^2 \\ \frac{b^2}{4} & \frac{-ab}{2} & \frac{b^2}{4}-\frac{a^2}{2} & \frac{ab}{2} & \frac{b^2}{4}+\frac{a^2}{2} & 0 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \end{bmatrix}$$

The Element stiffness matrix [K] :

$$[K] = \frac{2\pi E}{3b}\begin{bmatrix} \frac{5b^2}{4} & 0 & \frac{-3b^2}{4} & 0 & \frac{b^2}{4} & 0 \\ 0 & \frac{b^2}{2} & \frac{ab}{2} & \frac{-b^2}{2} & \frac{-ab}{2} & 0 \\ \frac{-3b^2}{4} & \frac{ab}{2} & \frac{5b^2}{4}+\frac{a^2}{2} & \frac{-ab}{2} & \frac{b^2}{4}-\frac{a^2}{2} & 0 \\ 0 & \frac{-b^2}{2} & \frac{-ab}{2} & \frac{a^2+b^2}{2} & \frac{ab}{2} & -a^2 \\ \frac{b^2}{4} & \frac{-ab}{2} & \frac{b^2}{4}-\frac{a^2}{2} & \frac{ab}{2} & \frac{b^2}{4}+\frac{a^2}{2} & 0 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \end{bmatrix}$$

Simplifying,

$$[K] = 2\pi E \begin{bmatrix} \frac{5b}{12} & 0 & \frac{-b}{4} & 0 & \frac{b^2}{12} & 0 \\ 0 & \frac{b}{6} & \frac{a}{6} & -\frac{b}{6} & \frac{-a}{6} & 0 \\ \frac{-b}{4} & \frac{a}{6} & \frac{5b}{12}+\frac{a^2}{6b} & \frac{-a}{6} & \frac{b}{12}-\frac{a^2}{6b} & 0 \\ 0 & \frac{-b}{6} & \frac{-a}{6} & \frac{a^2}{3b}+\frac{b}{6} & \frac{a}{6} & -\frac{a^2}{3b} \\ \frac{b}{12} & \frac{-a}{6} & \frac{b}{12}-\frac{a^2}{6b} & \frac{a}{6} & \frac{b}{12}+\frac{a^2}{6b} & 0 \\ 0 & 0 & 0 & -\frac{a^2}{3b} & 0 & \frac{a^2}{3b} \end{bmatrix}$$

Q II) To show row (2), (4) & (6) vanishes and row (1), (3) & (5) not vanishes. Why?

(i) For row (2), (4) & (6)

On keen observing the stiffness matrix, the addition of row (2), (4) & (6) is zero (vanishes).

$$Row(2) + Row(4) + Row(6) = \begin{bmatrix} 0 & \frac{b}{6} & \frac{a}{6} & \frac{-b}{6} & \frac{-a}{6} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-b}{6} & \frac{-a}{6} & \frac{a^2}{3b}+\frac{b}{6} & \frac{a}{6} & \frac{-a^2}{3b} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \frac{-a^2}{3b} & 0 & \frac{a^2}{3b} \end{bmatrix}$$

$$Row(2) + Row(4) + Row(6) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rigid body motion is achieved in z direction.
We get zero internal resistance forces to the prescribed motion, which delineates the rigid body motion in z direction.

ie. $[U^e] = [U_1 \ w_1 \ U_2 \ w_2 \ U_3 \ w_3]^T = [0 \ 1 \ 0 \ 1 \ 0 \ 1]^T$

(ii) For row (1), (3) & (5)

On keen observing the stiffness matrix, the addition of row (1), (3) & (5) is not zero.

$$Row(1) + Row(3) + Row(5) = \left[\frac{5b^2}{12} \quad 0 \quad \frac{-b}{4} \quad 0 \quad \frac{b}{12} \quad 0\right] + \left[\frac{-b}{4} \quad \frac{a}{6} \quad \frac{a^2}{6b}+\frac{5b}{12} \quad \frac{-a}{6} \quad \frac{b}{12}-\frac{a^2}{6b} \quad 0\right] + \left[\frac{b}{12} \quad \frac{-a}{6} \quad \frac{b}{12}-\frac{a^2}{6b} \quad \frac{a}{6} \quad \frac{b}{12}+\frac{a^2}{6b} \quad 0\right]$$

$$Row(1) + Row(3) + Row(5) \neq [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

So, the rigid body motion is not achieved in r direction.

Q III) The consistent force vector f^e.

$$[b] = [0, -q]^T$$

The body force (f_b) –

$$[f_b] = 2\pi \iint_A [N]^T \begin{Bmatrix} R_b \\ Z_b \end{Bmatrix} r \, dr \, dz.$$

$[f_b]$ is also evaluated by centroid of the element and $\bar{R_b}$ is radially directed body force.

$$\therefore, \quad [f_b] = \frac{2\pi \bar{r} A}{3} \begin{bmatrix} \bar{R_b} \\ Z_b \end{bmatrix} \qquad \text{where, } [N^q]^T = \begin{bmatrix} N_i^e & 0 \\ 0 & N_i^e \end{bmatrix}$$

ie,

$$[f_b] = \frac{2\pi \bar{r} A}{3} \begin{bmatrix} R_{b1} \\ Z_{b1} \\ R_{b2} \\ Z_{b2} \\ R_{b3} \\ Z_{b3} \end{bmatrix}$$

Substituting the values,

$$\cancel{b \neq [0\ \ -\rho g]^T} \quad b = [\bar{R}_b \ \ z_b]^T = [0 \ \ -\rho g]^T$$

$$\bar{r} = \frac{2a}{3}, \quad A = \frac{ab}{2}$$

$$[F_b] = \frac{2\pi}{3} \times \frac{2a}{3} \times \frac{ab}{2} \begin{bmatrix} 0 \\ -\rho g \\ 0 \\ -\rho g \\ 0 \\ -\rho g \end{bmatrix}$$

$$[F_b] = 2\pi \left[0 \ \ -\frac{\rho g a^2 b}{9} \ \ 0 \ \ -\frac{\rho g a^2 b}{9} \ \ 0 \ \ -\frac{\rho g a^2 b}{9} \right]^T$$

$$\boxed{[F_b] = -\frac{2\pi \rho g a^2 b}{9} \left[0 \ \ 1 \ \ 0 \ \ 1 \ \ 0 \ \ 1 \right]^T}$$

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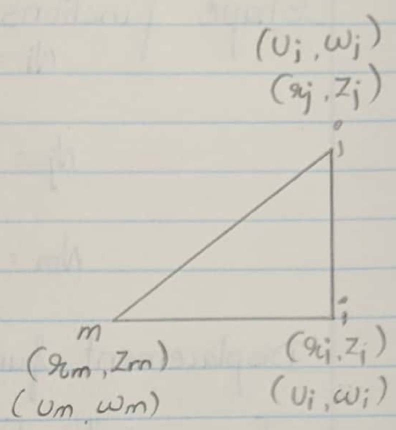
Reference :- 'A First Course in the Finite Element method' by Daryl L. Logan

Ref 1

Axisymmetric Elements - Derivation of Stiffness Matrix :



Step(1) Element Type -

An axisymmetric solid, along with a triangular element.
The element has 3 nodes with 2 degrees of freedom per node.

Step(2) Displacement functions

$$u(r,z) = a_1 + a_2 r + a_3 z$$
$$w(r,z) = a_4 + a_5 r + a_6 z$$

where, ~~As~~ total number of a_i's ie displacement functions is the same as the total number of degrees of freedom for the element.

Displacment Matrix -

$$\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} a_1 + a_2 r + a_3 z \\ a_4 + a_5 r + a_6 z \end{bmatrix} = \begin{bmatrix} 1 & r & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r & z \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{bmatrix}^T$$

ie, $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix}^{-1} \begin{bmatrix} u_i \\ u_j \\ u_m \end{bmatrix}$ & $\begin{bmatrix} a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix}^{-1} \begin{bmatrix} w_i \\ w_j \\ w_m \end{bmatrix}$

After performing inversion operation

$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \left\{ \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{bmatrix} u_i \\ u_j \\ u_m \end{bmatrix} \right.$ & $\begin{bmatrix} a_4 \\ a_5 \\ a_6 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{bmatrix} w_i \\ w_j \\ w_m \end{bmatrix}$

where, $\alpha_i = r_j z_m - r_m z_j$ $\quad \beta_i = z_j - z_m$ $\quad \gamma_i = r_m - r_j$

$\alpha_j = r_m z_i - r_i z_m$ $\quad \beta_j = z_m - z_i$ $\quad \gamma_j = r_i - r_m$

$\alpha_m = r_i z_j - r_j z_i$ $\quad \beta_m = z_i - z_j$ $\quad \gamma_m = r_j - r_i$

Ref 2

Shape functions -

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i r + \gamma_i z)$$

$$N_j = \frac{1}{2A}(\alpha_j + \beta_j r + \gamma_j z)$$

$$N_m = \frac{1}{2A}(\alpha_m + \beta_m r + \gamma_m z)$$

Displacement function -

$$\psi = \begin{bmatrix} U(r,z) \\ \omega(r,z) \end{bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{bmatrix}$$

ie, $\{\psi\} = [N][d]$

step(3) Strain & stress :

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{r} & 1 & \frac{z}{r} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

on simplifying -

$$[\varepsilon] = \frac{1}{2A} \begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \left(\frac{\alpha_i + \beta_i r + \gamma_i z}{r}\right) & 0 & \frac{\alpha_j + \beta_j r + \gamma_i z}{r} & 0 & \frac{\alpha_m + \beta_m r + \gamma_m z}{r} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \begin{bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{bmatrix}$$

ie,

$$[\varepsilon] = [\bar{B}][d]$$

$$\text{---}\ldots \quad [\bar{B}] = [\bar{B}_i \quad \bar{B}_j \quad \bar{B}_m]$$

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Ref 3

where,
$$[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \frac{\alpha_i + \beta_i r + \gamma_i z}{r} & 0 \\ \gamma_i & \beta_i \end{bmatrix}$$

stress -
$$[\sigma] = [D][B][d]$$

step (4) Element stiffness Matrix.

The stiffness matrix (K) :
$$[K] = \iiint_V [B]^T[D][B]\, dV$$
$$[K] = 2\pi \iint_A [B]^T[D][B]\, r\, dr\, dz$$

As [B] is function of r & z. $\therefore$ [K] is function of r & z.

To evaluate K, we can many ~~ways~~/methods such as

(i) Numerical Integration.

(ii) Explicit multiplication & term by term ~~multiplication~~ integration.

(iii) Evaluate B by centroid point $(\bar{r}, \bar{z})$ of the element.

ie,

By centroid -
$$[K] = 2\pi \bar{r} A [B]^T[D][B]$$

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