On "Isoparametric representation"

Assignment 4.1

A 3-node straight bar element is defined by 3 nodes: 1, 2 and 3 with axial coordinates x_1, x_2 and x_3 respectively as illustrated in figure below. The element has axial rigidity EA, and length $l = x_1 - x_2$. The axial displacement is u(x). The 3 degrees of freedom are the axial node displacement u_1, u_2 and u_3. The isoparametric definition of the element is

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} \tag{7.1}$$

in which $N^e_i(\xi)$ are the shape functions of a three bar element. Node 3 lies between 1 and 2 but is not necessarily at the midpoint $x = l/2$. For convenience define,

$$x_1 = 0 \qquad x_2 = l \qquad x_3 = (\tfrac{1}{2} + \alpha)l \tag{7.2}$$

where $-½ < \alpha < ½$ characterizes the location of node 3 with respect to the element center. If $\alpha=0$ node 3 is located at the midpoint between 1 and 2.

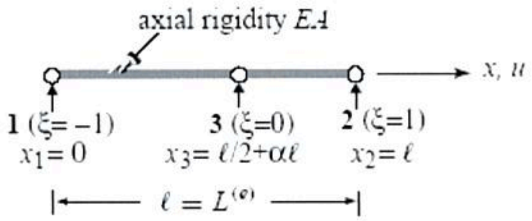

Figure.- The three-node bar element in its local system

1. From (7.2) and the second equation of (7.1) get the Jacobian $J = dx/d\xi$ in terms of l, α and ξ. Show that,

- if $-¼ < \alpha < ¼$ then $J > 0$ over the whole element $-1 \leq \xi \leq 1$.
- if $\alpha = 0$, $J = l/2$ is a constant over the element.

2. Obtain the 1x3 strain displacement matrix **B** relating $e = du/dx = \mathbf{B}\mathbf{u}^e$ where $\mathbf{u}^e$ is the column 3-vector of the node displacement u_1, u_2 and u_3. The entries of **B** are functions of l, α and ξ.

Hint: $\mathbf{B} = d\mathbf{N}/dx = J^{-1}d\mathbf{N}/d\xi$, where $\mathbf{N} = [\ N_1\ N_2\ N_3\ ]$ and J comes from item a).

(1)

Assignment 4 CSMD

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(4.1) three noded bar

$\varepsilon_1 = -1$ $\quad$ $\varepsilon_3 = 0$ $\quad$ $\varepsilon_2 = 1$

$x_1 = 0$ $\quad$ $x_3 = \frac{l}{2} + \alpha l$ $\quad$ $x_2 = l$

The shape functions for this element can be obtained using.

$$N_j = \prod_{\substack{i=1 \\ i \neq j}}^{3} \frac{(\varepsilon_i - \varepsilon)}{(\varepsilon_i - \varepsilon_j)} \qquad \text{(for 3 nodes)}$$

$$N_1 = \underbrace{\frac{(\varepsilon_2 - \varepsilon)}{(\varepsilon_2 - \varepsilon_1)}}_{j=1 \;\; i=2} \underbrace{\frac{(\varepsilon_3 - \varepsilon)}{(\varepsilon_3 - \varepsilon_1)}}_{j=1 \;\; i=3} = \frac{(1 - \varepsilon)}{[1 - (-1)]} \frac{(0 - \varepsilon)}{(-(-1))} = \frac{\varepsilon}{2}(1 - \varepsilon)$$

where $\begin{cases} \varepsilon_1 = -1 \\ \varepsilon_2 = 1 \\ \varepsilon_3 = 0 \end{cases}$ similarly for N_2 and N_3

$$N_2 = \frac{(\varepsilon_1 - \varepsilon)}{(\varepsilon_1 - \varepsilon_2)} \frac{(\varepsilon_3 - \varepsilon)}{(\varepsilon_3 - \varepsilon_2)} = \frac{\varepsilon}{2}(\varepsilon + 1)$$

$$N_3 = \frac{\varepsilon_1 - \varepsilon}{\varepsilon_1 - \varepsilon_3} \frac{\varepsilon_2 - \varepsilon}{\varepsilon_2 - \varepsilon_3} = 1 - \varepsilon^2$$

to represent N_1, N_2 and N_3 in cartesian coordinates we need to find the Jacobian $\frac{dx}{d\varepsilon}$

(2)

since $\quad x = x_1 N_1 + x_2 N_2 + x_3 N_3$

with $\quad x_1 = 0, \quad x_2 = \ell \quad$ & $\quad x_3 = \ell\left(\frac{1}{2} + \alpha\right)$

we set

$$x = 0 + \ell \frac{\varepsilon}{2}(\varepsilon + 1) + \ell\left(\frac{1}{2} + \alpha\right)(1 - \varepsilon^2)$$

differentiating w.r.t. ε

$$\frac{dx}{d\varepsilon} = \frac{d}{d\varepsilon}\left[\frac{\ell}{2}(\varepsilon^2 + \varepsilon) + \ell\left(\frac{1}{2} + \alpha\right) - \ell\left(\frac{1}{2} + \alpha\right)\varepsilon^2\right]$$

$$= \frac{\ell}{2}(2\varepsilon + 1) - 2\ell\left(\frac{1}{2} + \alpha\right)\varepsilon$$

$$= \cancel{\varepsilon\ell} + \frac{\ell}{2} - \cancel{\ell\varepsilon} - 2\alpha\varepsilon\ell$$

$$= \frac{\ell}{2} - 2\alpha\varepsilon\ell = \boxed{\frac{\ell}{2}(1 - 4\alpha\varepsilon)}$$

the Jacobian $\frac{dx}{d\varepsilon} > 0$ if. $\quad (1 - 4\alpha\varepsilon) > 0$

$$1 > |4\alpha\varepsilon|$$

$$\frac{1}{4} > |\alpha||\varepsilon|$$

$$-\frac{1}{4} < \alpha < \frac{1}{4}$$

since $|\varepsilon| \leq 1$ always

for $\alpha = 0$ the $\frac{dx}{d\varepsilon} = \frac{\ell}{2}$, a constant.

(3)

Find the strain displacement matrix $\bar{\bar{B}}$ relating $e = du/dx = \bar{\bar{B}} \bar{u}_e$

$$\left[\bar{\bar{B}} = \frac{d\bar{N}}{dx} = J^{-1} \frac{d\bar{N}}{d\varepsilon} \right]$$

<u>solution</u>

$$B_i = \frac{dN_i}{dx} = \frac{dN_i}{d\varepsilon} \cdot \frac{d\varepsilon}{dx} = J^{-1} \frac{dN_i}{d\varepsilon}$$

using the previous result:

$$J^{-1} = \left[\frac{l}{2} (1 - 4\alpha\varepsilon) \right]^{-1} = \frac{2}{l(1 - 4\alpha\varepsilon)}$$

$$\bar{N} = \left[\frac{dN_1}{d\varepsilon} \quad \frac{dN_2}{d\varepsilon} \quad \frac{dN_3}{d\varepsilon} \right]$$

$$= \left[\frac{1}{2} - 2\varepsilon \quad \varepsilon + \frac{1}{2} \quad -2\varepsilon \right]$$

therefore $\bar{\bar{B}} = J^{-1} \bar{N}$ becomes:

$$\bar{\bar{B}} = \frac{1}{l(1 - 4\alpha\varepsilon} \left[1 - 4\varepsilon \quad 2\varepsilon + 1 \quad -4\varepsilon \right]$$

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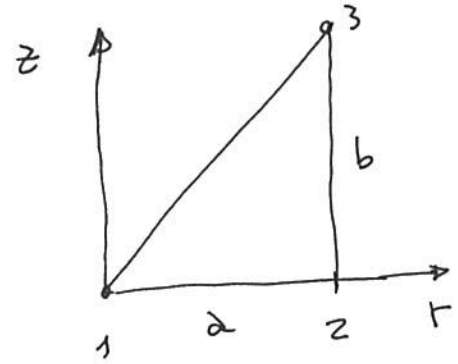
4.2 compute the entries of the element stiffness matrix K^e for an axisymmetric triangle. Consider $\nu = 0$ (Poisson's ratio) and a stress-strain matrix $\bar{D}$

$$\bar{D} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

E = Young's Mod.



working with polar coordinates (r, θ, z) symmetry around the z axis

the stiffness matrix is defined as

$$\bar{\bar{K}} = \int_\Omega \bar{\bar{B}}^T \bar{\bar{D}} \bar{\bar{B}} \, d\Omega$$

to compute the volume of a rotation solid we use

$$K_{ij} = 2\pi \int \bar{\bar{B}}^T \bar{\bar{D}} \bar{\bar{B}} \, r \, dr \, dz$$

where

$$B_i = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 \\ 0 & \frac{\partial N_i}{\partial z} \\ \frac{N_i}{r} & 0 \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} \end{bmatrix} \qquad i = \{1, 2, 3\}$$

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using the definitions of the shape functions
$$N_i = \frac{a_i + b_i r + c_i z}{2A}$$

$$a_i = r_j z_m - r_m z_j$$
$$b_i = z_j - z_m$$
$$c_i = r_m - r_j$$
$$A = \frac{ab}{2}$$

we set
$$N = \frac{1}{ab}\begin{bmatrix} ab - br & br - az & az \end{bmatrix}$$

$$\frac{\partial N}{\partial r} = \frac{1}{ab}\begin{bmatrix} -b & b & 0 \end{bmatrix}$$
$$\frac{\partial N}{\partial z} = \frac{1}{ab}\begin{bmatrix} 0 & -a & a \end{bmatrix}$$

therefore

$$\overline{\overline{B}} = \frac{1}{ab}\begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{ab}{r} - b & 0 & b - \frac{az}{r} & 0 & \frac{az}{r} & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

The product $\overline{B}^T \overline{\overline{D}}\, \overline{B}$ is dependent on r and z therefore the integration is not trivial. it can be done element by element

an alternative is to fix r and z at their centroid values
$$r = \frac{1}{3}\sum_1^3 r_i = \frac{2a}{3}$$

$$z = \frac{1}{3}\sum_1^3 z_i = \frac{b}{3}$$

(6) this way we can evaluate directly

$$K = \int 2\pi\, \bar{\bar{B}}^T \bar{\bar{D}}\, \bar{\bar{B}}\, r\, dr\, dz = 2\pi\, \bar{\bar{B}}_c^T \bar{\bar{D}}_c \bar{\bar{B}}_c\, r_c\, A$$

c = centroid.

$$K = \frac{E\pi}{ab}\begin{bmatrix} \frac{5b^2}{4} & 0 & -\frac{3b^2}{4} & 0 & \frac{b^2}{4} & 0 \\ & \frac{b^2}{2} & \frac{ab}{2} & -\frac{b^2}{2} & -\frac{ab}{2} & 0 \\ & & \frac{5b^2+2a^2}{4} & -\frac{ab}{2} & \frac{b^2-2a^2}{4} & 0 \\ & & & \frac{2a^2+b^2}{2} & \frac{ab}{2} & -a^2 \\ & \text{Symm} & & & \frac{b^2+2a^2}{4} & 0 \\ & & & & & a^2 \end{bmatrix}$$

(2) Rows 2, 4 and 6 add up to zero.
this is the equilibrium in the z direction
has to be guaranteed by the stiffness
matrix as the problem is not constrained
on the other hand, rows 1, 3 and 5
do not sum up to zero. this is
because the axial symmetry guarantees
equilibrium.

7

3 Compute the vector of body forces

$$\bar{f} = -2\pi \int_V N \bar{b}\, r \, dr\, dz$$

again, using the centroid values to avoid integration

$$\bar{f} = -2\pi \bar{N}(r_c, z_c) \begin{bmatrix} b_r \\ b_z \end{bmatrix} r_c A$$

$$\bar{f} = -\frac{2}{9}\pi a^2 b \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} = -\frac{2}{9}\pi a^2 b g \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$