Computational Structural Mechanics and Dynamics

Assignment 4

Structures of Revolution

By

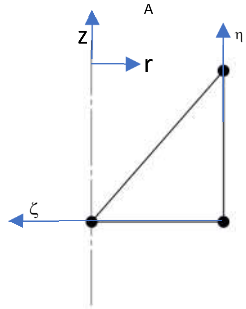
Domingo Eugenio Cattoni Correa

Master in numerical method in engineering

Assignment 4.1:

1. The stiffness matrix of an axisymmetric triangle will be:

$$K_e = 2\pi \int_A r B^T C B dA$$



But, if it is used natural coordinate, then, the stiffness matrix will take the next expression:

$$K_e = 2\pi \int_0^1 \int_0^{1-\eta} r B^T C B |J| d\zeta d\eta$$

Where:

$$C = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \quad D' = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ 1 & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \quad N = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix}$$

$B = D' * N$

$N_1 = \zeta,\ N_2 = 1 - \zeta - \eta,\ \text{and } N_3 = \eta$

Before continuing, it will be necessary to interpolate the radius as $r = \sum_i r_i N_i$, and the matrix B will be split in two matrices, called, D and H, such that B = D*H. It will use natural coordinate.

$$D = \begin{bmatrix} \frac{\partial \zeta}{\partial r} & \frac{\partial \eta}{\partial r} & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \zeta}{\partial z} & \frac{\partial \eta}{\partial z} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{\partial \zeta}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial r} & \frac{\partial \eta}{\partial r} & 0 \end{bmatrix} \quad H = \begin{bmatrix} \frac{\partial N_1}{\partial \zeta} & \frac{\partial N_2}{\partial \zeta} & \frac{\partial N_3}{\partial \zeta} & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial \zeta} & \frac{\partial N_2}{\partial \zeta} & \frac{\partial N_3}{\partial \zeta} \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} \\ \frac{N_1}{r} & \frac{N_2}{r} & \frac{N_3}{r} & 0 & 0 & 0 \end{bmatrix}$$

The matrix D contains the component of the inverse of the Jacobian matrix.

$$J=\begin{bmatrix}\frac{\partial r}{\partial \zeta} & \frac{\partial z}{\partial \zeta}\\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta}\end{bmatrix} \quad J^{-1}=\begin{bmatrix}\frac{\partial \zeta}{\partial r} & \frac{\partial \eta}{\partial r}\\ \frac{\partial \zeta}{\partial z} & \frac{\partial \eta}{\partial z}\end{bmatrix}$$

Results:

$$J=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix} \quad J^{-1}=\begin{bmatrix}\frac{1}{a} & 0\\ 0 & \frac{1}{b}\end{bmatrix}$$

$$D=\begin{bmatrix}\frac{1}{a} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{b} & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & \frac{1}{b} & \frac{1}{a} & 0 & 0\end{bmatrix} \quad H=\begin{bmatrix}1 & -1 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & -1 & 1\\ \frac{\zeta}{a(\zeta-1)} & -\frac{\zeta+\eta-1}{a(\zeta-1)} & \frac{\eta}{a(\zeta-1)} & 0 & 0 & 0\end{bmatrix}$$

And the matrix B will be:

$$B=\begin{bmatrix}\frac{1}{a} & -\frac{1}{a} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -\frac{1}{b} & \frac{1}{b}\\ \frac{\zeta}{a(\zeta-1)} & -\frac{\zeta+\eta-1}{a(\zeta-1)} & \frac{\eta}{a(\zeta-1)} & 0 & 0 & 0\\ 0 & -\frac{1}{b} & \frac{1}{b} & \frac{1}{a} & -\frac{1}{a} & 0\end{bmatrix}$$

Finally, and using Gauss quadrature as integration method, the stiffness matrix is:

$$K_e=\int_0^1\int_0^{1-\eta} rB^TCB\left|J\right|d\zeta d\eta=\int_0^1\int_0^{1-\eta} F(\zeta,\eta)d\zeta d\eta\approx F(\zeta=1/3,\eta=1/3)$$

$$K_e=E\begin{bmatrix}-\frac{5b}{6} & \frac{b}{2} & -\frac{b}{6} & 0 & 0 & 0\\ \frac{b}{2} & -\frac{2}{3}(\frac{5b}{4}+\frac{a^2}{2b}) & -\frac{2}{3}(\frac{b}{4}-\frac{a^2}{2b}) & \frac{a}{3} & -\frac{a}{3} & 0\\ -\frac{b}{6} & -\frac{2}{3}(\frac{b}{4}-\frac{a^2}{2b}) & -\frac{2}{3}(\frac{b}{4}+\frac{a^2}{2b}) & -\frac{a}{3} & \frac{a}{3} & 0\\ 0 & \frac{a}{3} & -\frac{a}{3} & -\frac{b}{3} & \frac{b}{3} & 0\\ 0 & -\frac{a}{3} & \frac{a}{3} & \frac{b}{3} & -\frac{2}{3}(\frac{b}{2}+\frac{a^2}{2b}) & \frac{2a^2}{3b}\\ 0 & 0 & 0 & 0 & \frac{2a^2}{3b} & -\frac{2a^2}{3b}\end{bmatrix}$$

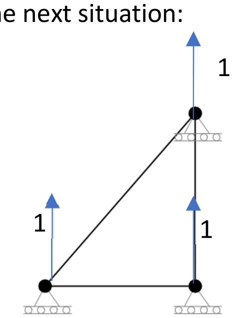
Remark 1: It has considered the unknown vector U is formed as:

$$U=\begin{bmatrix} U_{r1} \\ U_{r2} \\ U_{r3} \\ U_{z1} \\ U_{z2} \\ U_{z3} \end{bmatrix}$$

2. It can be seen the summation of the third last column are equal to zero. As explanation of this problem, let it considered the next displacement vector:

$$U=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Graphically, this vector represents the next situation:

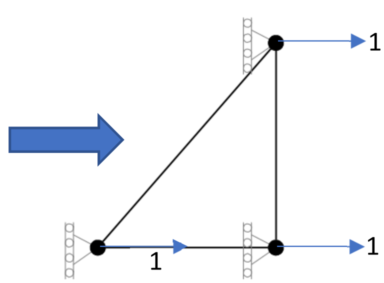


When it is computed the internal forces in order to obtain the reaction forces, then, it get.

$$KU=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and the deformation vector } \varepsilon=BU=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{, there is no doformation.}$$

It seems, that the system is subject to a rigid body motion in the z direction but is not true. Now, it is considered the next displacement vector U.

$$U=\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



When it is computed the internal forces in order to obtain the reaction forces, then it get.

$$KU=\begin{bmatrix} -\frac{b}{2} \\ -\frac{b}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

As a conclusion, it can be said that this type of element has a lack of stiffness in z direction, in order to avoid that, it will be necessary not only constrain the r direction, but also constrain a z direction of one of the nodes.

3. The element has been submitted to body force, more precisely the gravity force $b=\begin{bmatrix}0 & -g\end{bmatrix}$ then, the consistent force vector will be:

$$f^e=\int_0^1\int_0^{1-\eta} rN b^T |J| d\zeta d\eta=\begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{a^2bg}{12} \\ -\frac{a^2bg}{8} \\ -\frac{a^2bg}{8} \end{bmatrix} \quad f^e=\begin{bmatrix} f_r^e \\ f_z^e \end{bmatrix}$$

$$f_r^e=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ f_z^e=\begin{bmatrix} -\frac{a^2bg}{12} \\ -\frac{a^2bg}{8} \\ -\frac{a^2bg}{8} \end{bmatrix}$$