On "Plane stress problem" and "Linear Triangle"

Assignment 3.1

1. Compute the entries of $\mathbf{K}_e$ for the following plane stress triangle:

$$x_1 = 0,\ y_1 = 0,\ x_2 = 3,\ y_2 = 1,\ x_3 = 2,\ y_3 = 2$$

$$\boldsymbol{E} = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix},\quad h = 1$$

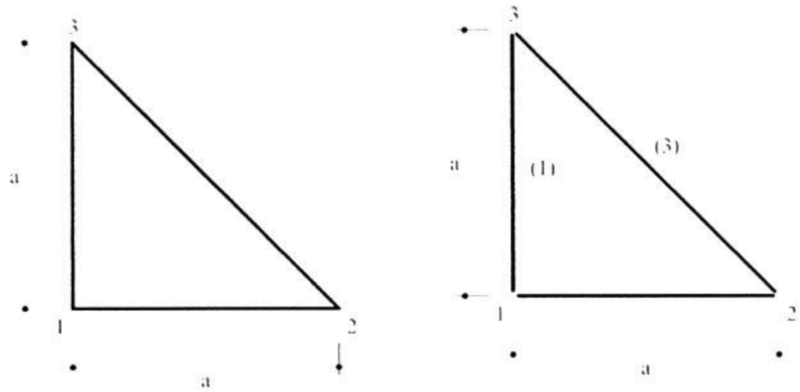
Partial result: $K_{11} = 18{,}75$, $K_{66} = 118{,}75$.

2. Show that the sum of the rows (and columns) 1, 3 and 5 of $\mathbf{K}^e$ as well as the sum of rows (and columns) 2, 4 and 6 must vanish, and explain why.

Assignment 3.2

Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Let's consider for simplicity $a = h = 1$. The material parameters are E,ν. Initially ν is set to zero. Two structural models are considered for this problem as depicted in the figure:

- A plane linear Turner triangle with the same dimensions.
- A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.



a) Calculate the stiffness matrix $\boldsymbol{K}^e$ for both models.

b) Is there any set of values for cross sections A_1=A_2=A and A_3=A' to make both stiffness matrix equivalent: $\boldsymbol{K}_{bar} = \boldsymbol{K}_{triangle}$? If not, which are these values to make them as similar as possible?

c) Why these two stiffness matrix are not equivalent? Find a physical explanation.

d) Solve question a) considering ν≠0 and extract some conclusions.

Note: To solve this assignment it's recommended to check the features of the linear triangle in presentation "CSMD_05_Linear_Triangle". Some comments will be given in the next class.

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The assignment must be submitted as a pdf file named **As3-Surname.pdf** to the CIMNE virtual center.

Assignment 3

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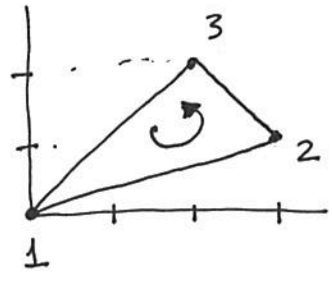
3.1 Compute K_e (element stiffness matrix) for the following plane stress triangle

P	X	Y
1	0	0
2	3	1
3	2	2



$$D = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}$$

$h = 1$

first we compute the area of the triangle.

$$A = \frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\ X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{bmatrix} = \frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix} = 2$$

the triangular shape functions ζ_i can be expressed in coordiates x, y of the cartesian plane using

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} 2A_{23} & y_{23} & x_{23} \\ 2A_{31} & y_{31} & x_{31} \\ 2A_{12} & y_{12} & x_{12} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

where $x_{jk} = x_j - x_k$ and $y_{jk} = y_j - y_k$ and A_{jk} is the area of the triangles $(0,0), (x_j\, y_j), (x_k, y_k)$

(2) the strain element matrix B is

$$\overline{\overline{B}} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 5 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

withe the element stiffness matrix being.

$$K^e = \int_{A^e} h \overline{\overline{B}}^T \overline{\overline{E}}\, \overline{\overline{B}}\, dA$$

since all the elements inside the integral are constant we can write

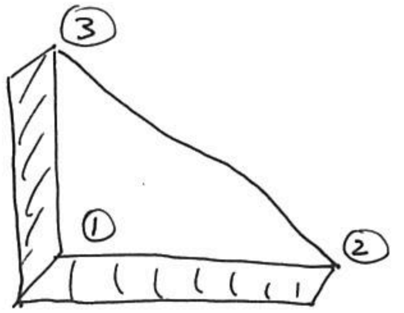
$$K^e = Ah\, \overline{\overline{B}}^T \overline{\overline{E}}\, \overline{\overline{B}} = \frac{1}{8}\begin{bmatrix} 150 & 75 & -100 & -50 & -50 & -25 \\ \cdot & 150 & 50 & 100 & -125 & -250 \\ \cdot & \cdot & 600 & -300 & -500 & 250 \\ \cdot & \cdot & \cdot & 600 & 350 & -700 \\ & \text{symm.} & \cdot & \cdot & 550 & -225 \\ & & \cdot & \cdot & \cdot & 950 \end{bmatrix}$$

We can verify that $K_{11} = 18.75$ and $K_{66} = 118.75$.

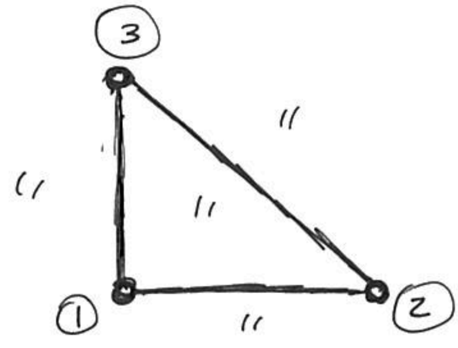
We also notice that the sum of all elements is zero in the rows and colums of K^e (that is $\sum_{i=1}^{6} K^e_{ij} =$ for every j and $\sum_{j=1}^{6} K_{ij} = 0$ for every i)

why? Assume $\bar{d} = [1, 1, 1 \dots 1]^T$ in $\overline{\overline{K}}^e \bar{d} = \bar{f}^e$. this is displacement of all the element as a rigid body the net force must be zero (1st Newton's law) and therefore the rows of K_{ij} must add to zero the same applies to columns $K_{ij} = K_{ji}$ $[\overline{\overline{K}} = \overline{\overline{K}}^T]$

3.2 Two models



solid triangle



three truss bars.

a)

- the stiffness matrix for the solid triangle was calculated previously. here we just update the values for area $A = \frac{a^2}{2} = 1/2$

P	X	Y
1	0	0
2	1	0
3	0	1

$$\bar{\bar{D}} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Elastic material matrix in plane stress

$\nu = 0$
$a = 1$
$h = 1$

$$\bar{\bar{D}} = \frac{E}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \qquad \bar{\bar{B}} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$K^e = \int h\, \bar{\bar{B}}^T \bar{\bar{D}}\, \bar{\bar{B}}\, dA = \frac{E}{2}\begin{bmatrix} 3/2 & 1/2 & -1 & -1/2 & -1/2 & 0 \\ \cdot & 3/2 & 0 & -1/2 & -1/2 & -1 \\ \vdots & \vdots & 1 & 0 & 0 & 0 \\ \vdots & \cdot & \cdots & 1/2 & 1/2 & 0 \\ Sym\ldots & & \cdots & \cdots & 1/2 & 0 \\ & & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

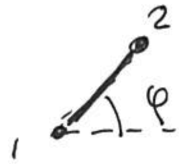
- the stiffness matrix for the three-truss element can be calculated by assembly, using compatibility & equilibrium, of the individual stiffness matrices using D.F.M.

the stiffness for each truss element is

$$K^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

$s = \sin \varphi$

$c = \cos \varphi$



Calculating K^e, $e = \{1, 2, 3\}$

and assembling we get

$$K = E \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ & -A & 0 & 0 & 0 & 0 \\ & & A+\alpha & -\alpha & -\alpha & \alpha \\ & & & \alpha & \alpha & -\alpha \\ & \text{Symm.} & & & \alpha & -\alpha \\ & & & & & A+\alpha \end{bmatrix}$$

where $\alpha = 0.3535\, A'$

b) there is no set of values of cross-sectional areas of the three-truss element that will make the K matrix equivalent to the solid triangle's K. the latter has many non-zero elements that are zero on the former. there is no scalar multiplier that can make them equal except zero.

c) the two physical systems are very different: the three-truss element can only have compression-tension stress along the bars. the hinges prevent any shear stresses. the solid triangle does have shear stresses and more complex deformations.

d) the Poisson Ratio ν describes contraction or expansion perpendicular to the load
for $\nu \neq 0$ the solid triangle stiffness matrix is

$$K^e = E \begin{bmatrix} 1+\frac{1-\nu}{2} & \nu+\frac{1-\nu}{2} & -1 & -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & -\nu \\ & 1+\frac{1-\nu}{2} & -\nu & -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & -1 \\ & & 1 & 0 & 0 & \nu \\ & & & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ & \text{symm.} & & & \frac{1-\nu}{2} & 0 \\ & & & & & 1 \end{bmatrix}$$

Horizontal stresses result in vertical deformation and vice versa. the overal stiffness matrix changes depending on the Poisson Ratio.
the truss-bar triangle is not affected by ν as the deformation perpendicular to the axial load do not affect the other truss elements.