Computational Structural Mechanics and Dynamics

Assignment 3

Plane stress problem

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Master in numerical method in engineering

Assignment 3.1:

In isotropic elastic materials (as well as in plasticity and viscoelasticity) it is convenient to use the so-called Lamé constants λ and μ instead of E and ν in the constitutive equations. Both λ and μ have the physical dimension of stress and are related to E and ν by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \mu = G = \frac{E}{2(1+\nu)}$$

1. It can be seen different steps in order to obtain E and υ in terms of Lame's parameters μ and λ:

$E\upsilon = (1+\upsilon)(1-2\upsilon)\lambda$ and $E=2\mu(1+\upsilon)$

$2\mu(1+\upsilon)\upsilon = (1+\upsilon)(1-2\upsilon)\lambda \rightarrow (1-2\upsilon)\lambda = 2\mu\upsilon \rightarrow \lambda = 2\mu\upsilon + 2\lambda\upsilon$

It is obtained $\rightarrow \upsilon = \frac{\lambda}{2(\mu+\lambda)}$

Taking and replacing $\upsilon = \frac{\lambda}{2(\mu+\lambda)}$ into $E=2\mu(1+\upsilon)$, then it gets $E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}$

Finally, $E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}$ and $\upsilon = \frac{\lambda}{2(\mu+\lambda)}$

2. The constitutive matrix for plane stress and strain are:

$$C_{plane\_stress} = \frac{E}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \quad C_{plane\_strain} = \frac{E(1-\upsilon)}{(1+\upsilon)(1-2\upsilon)}\begin{bmatrix} 1 & \frac{\upsilon}{1-\upsilon} & 0 \\ \frac{\upsilon}{1-\upsilon} & 1 & 0 \\ 0 & 0 & \frac{1-2\upsilon}{2(1-\upsilon)} \end{bmatrix}$$

Taking and replacing E and υ into C_{plane_stress} and C_{plane_strain}, then, it is obtained:

$$C_{plane\_stress} = \begin{bmatrix} \frac{4\mu(\mu+\lambda)}{2\mu+\lambda} & \frac{2\mu\lambda}{2\mu+\lambda} & 0 \\ \frac{2\mu\lambda}{2\mu+\lambda} & \frac{4\mu(\mu+\lambda)}{2\mu+\lambda} & 0 \\ 0 & 0 & \mu \end{bmatrix} \text{ and } C_{plane\_strain} = \begin{bmatrix} 2(\mu+\lambda) & \lambda & 0 \\ \lambda & 2(\mu+\lambda) & 0 \\ 0 & 0 & 2\mu+\lambda \end{bmatrix}$$

3. The constitutive matrix correspond to the plane strain problems can be split in two matrices, one matrix will contain only μ parameter and the other will contain λ.

$C_{plane\_strain} = E_{\lambda} + E_{\upsilon}$

$$E_{\lambda} = \lambda\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{\upsilon} = 2\mu\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Each matrix E, written above, can be expressed in terms of E and υ as:

$$E_{\lambda}=\frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)}\begin{bmatrix}2&1&0\\1&2&0\\0&0&2\end{bmatrix}\ E_{\upsilon}=\frac{2E}{2(1+\upsilon)}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Assignment 3.2:

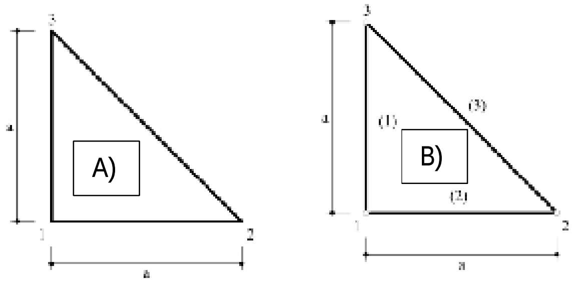

Figure 1: Two discrete structural models a) Triangle, b) Triangle formed by bars.

Parameters:
For simplicity it will consider a = 1 and h (thickness) = 1. The material parameters are E and υ, initially υ = 0. It will consider plane stress problem.

1. Stiffness matrix for each discrete model is:

Stiffness matrix for the discrete structure modelled with a triangle.

$$K^e=\int_{\Omega}hB^TEBd\Omega$$

Where

$$B=\frac{1}{2A}\begin{bmatrix}y_{23}&0&y_{31}&0&y_{12}&0\\0&x_{32}&0&x_{13}&0&x_{21}\\x_{32}&y_{23}&x_{13}&y_{31}&x_{21}&y_{12}\end{bmatrix},\ A=\frac{a^2}{2}=\frac{1}{2}$$

$y_{ij}=y_i-y_j$ and $x_{ij}=x_i-x_j$

As the problem is a plane stress problem, the constitutive matrix is:

$$E_{plane\_stress}=\frac{E}{1-\upsilon^2}\begin{bmatrix}1&\upsilon&0\\\upsilon&1&0\\0&0&\frac{1-\upsilon}{2}\end{bmatrix}\text{ considering }\upsilon=0\ E_{plane\_stress}=E\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{2}\end{bmatrix}$$

As B, E and h are constant over the triangle, they can be taken off of the integral. Finally, the stiffness matrix is:

$$K^e=\frac{1}{4A^2}hB^TE_{plane\_stress}B\int_{\Omega}d\Omega=\frac{1}{4A}hB^TE_{plane\_stress}B\ \rightarrow K^e=\frac{E}{4}\begin{bmatrix}3&1&-2&-1&-1&0\\1&3&0&-1&-1&-2\\-2&0&2&0&0&0\\-1&-1&0&1&1&0\\-1&-1&0&1&1&0\\0&-2&0&0&0&2\end{bmatrix}$$

$$K_{triangle} = \frac{E}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Stiffness matrix for the discrete structure modelled with bars.

Taking into account the first assignment, the stiffness matrix can be formed considering the stiffness matrix of each bar separately and then, the assembly process can be applied. In this case A_1=A_2=A.

$$K_{bar\_1} = EA\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$K_{bar\_2} = EA\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_{bar\_3} = \frac{\sqrt{2}EA_3}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 \\ 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \\ 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \\ 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

Now, $K_{bars} = K_{bar\_1} + K_{bar\_2} + K_{bar\_3}$

$$K_{bars} = E\begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A+0.25\sqrt{2}A_3 & -0.25\sqrt{2}A_3 & -0.25\sqrt{2}A_3 & 0.25\sqrt{2}A_3 \\ 0 & 0 & -0.25\sqrt{2}A_3 & 0.25\sqrt{2}A_3 & 0.25\sqrt{2}A_3 & -0.25\sqrt{2}A_3 \\ 0 & 0 & -0.25\sqrt{2}A_3 & 0.25\sqrt{2}A_3 & 0.25\sqrt{2}A_3 & -0.25\sqrt{2}A_3 \\ 0 & -A & 0.25\sqrt{2}A_3 & -0.25\sqrt{2}A_3 & -0.25\sqrt{2}A_3 & A+0.25\sqrt{2}A_3 \end{bmatrix}$$

2. It can be seen that both matrices are different, it is difficult to find some values A_1=A_2 and A_3 to make both stiffness matrix equivalent.

$$K_{triangle} = \frac{E}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$K_{bars} = E\begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A+0.25\sqrt{2}A_3 & -0.25\sqrt{2}A_3 & -0.25\sqrt{2}A_3 & 0.25\sqrt{2}A_3 \\ 0 & 0 & -0.25\sqrt{2}A_3 & 0.25\sqrt{2}A_3 & 0.25\sqrt{2}A_3 & -0.25\sqrt{2}A_3 \\ 0 & 0 & -0.25\sqrt{2}A_3 & 0.25\sqrt{2}A_3 & 0.25\sqrt{2}A_3 & -0.25\sqrt{2}A_3 \\ 0 & -A & 0.25\sqrt{2}A_3 & -0.25\sqrt{2}A_3 & -0.25\sqrt{2}A_3 & A+0.25\sqrt{2}A_3 \end{bmatrix}$$

If $A = \frac{3E}{4}$ and $A_3 = \frac{\sqrt{2}E}{2}$ some components of both matrices are equal.

3. First of all, it can be said that the differences between these two matrices lies in the fact that bar elements have 1 degree of freedom per node while triangular elements have two degrees of freedom per node, in other words, bars have axial stiffness and triangles have stiffness in x and y direction. Bar elements are 1D elements while triangle elements are 2D elements. Bar elements are able to support axials stresses and triangular elements can support not only axials stresses, but also, shear stresses.
Second, each zero component of the stiffness matrix does not contribute in the stiffness of the system, then, the system is less able to support loads.
Finally, physically speaking the internal points of the triangle, that form its area, contribute to the stiffness of the system. While, the areas of the bars only contribute to the stiffness of the system.

4. Considering the fact that $\upsilon \neq 0$ then, the triangle stiffness matrix will be:

$$K_{triangle} = \frac{E}{4(1-\upsilon^2)}\begin{bmatrix} 3-\upsilon & 1+\upsilon & -2 & \upsilon-1 & \upsilon-1 & -2\upsilon \\ \upsilon+1 & 3-\upsilon & -2\upsilon & \upsilon-1 & \upsilon-1 & -2 \\ -2 & -2\upsilon & 2 & 0 & 0 & 2\upsilon \\ \upsilon-1 & \upsilon-1 & 0 & 1-\upsilon & 1-\upsilon & 0 \\ \upsilon-1 & \upsilon-1 & 0 & 1-\upsilon & 1-\upsilon & 0 \\ -2\upsilon & -2 & 2\upsilon & 0 & 0 & 2 \end{bmatrix}$$

It can be seen that some components that were zero before, now are not. This contributes in the stiffness of the system.