CSMD Assignment 10

DYNAMICS

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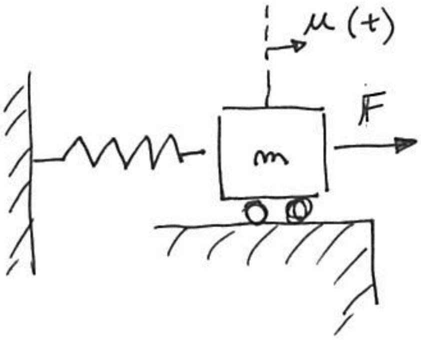

Given the mass m subject to the force F (constant). Describe the effect of F on $u(t)$ [displacement relative to equilibrium] and the natural vibration frequency.

Solution

Using Newton's second law, the sum of forces is equal to the acceleration of the body.

$$\sum F_{i\,ext} = m\,a = m\,\frac{d^2u}{dt^2} \qquad (1)$$

$$\sum F_{i\,ext} = -k\,u + F \qquad (2)$$

from (1) and (2) it follows that

$$-k\,u + F = m\,\frac{d^2u}{dt^2}$$

or

$$\frac{d^2u}{dt^2} + \frac{k}{m}\,u + \frac{F}{m} = 0 \qquad (3)$$

which is a second order O.D.E.
the solution is given by a particular solution and the null space of the equation [solution to the homogeneous equation]

- using the solution (for the null space)

$$u = A\cos(wt+\varphi) + B\sin(wt+\varphi)$$

using $u(0)=0$, $\phi=0$ we get

$$\frac{d^2u}{dt} = -Aw^2\cos(wt+\varphi) - Bw^2\sin(wt+\varphi)$$

$$= -Aw^2\cos(wt) - Bw^2\sin(wt) \qquad (4)$$

Substitute into (3) and we obtain [with $F=0$]

$$-w^2\left[A\cos(wt) + B\sin(wt)\right] + \frac{k}{m}\left[A\cos wt + B\sin(wt)\right] = 0$$

since $u(0)=0$ $A=0$ (otherwise $u\neq 0$ for $t=0$)

$$\Rightarrow \quad u(t) = B\sin(wt) \quad ; \quad w = \sqrt{k/m}$$

(we could have chosen $\varphi\neq 0$ and the solution would be different)

- a particular solution is given by

$$u_p = \frac{F}{k} \quad \text{as} \quad u' = u'' = 0$$

so (3) becomes $\frac{k}{m}\left(\frac{F}{k}\right) + \frac{F}{m} = 0$

which is true.

therefore, the general solution is

$$u(t) = u_p + u_g = \frac{F}{k} + B\sin(wt) \qquad (5)$$

since $B = -F/k$

$$u = \frac{F}{k}\left[1 - \sin\left(\sqrt{k/m}\, t\right)\right] \quad (6)$$

the force F does not affect the natural frequency of the system, only its amplitude.

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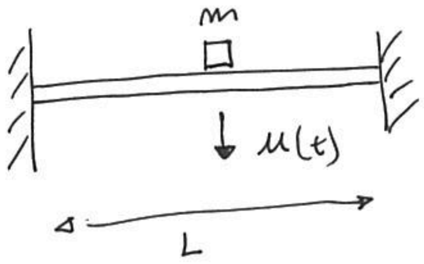

Neglecting the mass of the bar, find the natural frequency in terms of m, L, E (of the bar) and A (of the bar).

Solution

Since the system has 1 degree of freedom (vertical displacement of m) we can analyze it as a vertical oscillator

the displacement in the middle of the beam is

$$u = \frac{FL^3}{16EA^2} = \frac{mgL^3}{16EA^2} \quad (1) \quad (\text{where } F = mg)$$

Balancing forces assuming the force exerted by the bar is linear with the displacement u then $Ku = mg$

$$K \frac{mgL^3}{16EA^2} = mg$$

$$K = \frac{16EA^2}{L^3} \quad (2)$$

the frequency of vibration is then given by:

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{16 E A^2}{m L^3}} = \frac{4A}{L}\sqrt{\frac{E}{m L^2}} \quad (3)$$

(3) use the expression $m = \int \bar{N}^T \bar{N} \rho \, dV$ (1) to derive the mass matrix

$$m = \frac{\rho A L}{6}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad (2)$$

Using the same shape function to derive the stiffness matrix, we obtain a consistent matrix. For a 2-noded bar with 1 D.O.F. (x) the shape functions are

$$N_1 = 1 - \varepsilon \quad \text{and} \quad N_2 = \varepsilon \quad (3)$$

$$\bar{N} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \qquad \bar{N}^T = [N_1 \ N_2]$$

$$\bar{N}^T \bar{N} = (N_1 \ N_2)\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix} \quad \text{(outer product)}$$

$$\Rightarrow \quad m = \frac{\rho A}{L^2}\int_0^L \begin{bmatrix} (L-x)^2 & x(L-x) \\ x(L-x) & x^2 \end{bmatrix} dx \quad (4) \qquad dx = L\,d\varepsilon$$

$$m = \frac{\rho A L}{6}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Since $\int_0^L (L-x)^2\, dx = \int_0^L L^2 - 2Lx - x^2\, dx$

$$= L^3 - \frac{2}{2}L^3 + \frac{L^3}{3} = \frac{L^3}{3}$$

and $\int_0^L x(L-x)\, dx = \int_0^L xL - x^2\, dx$

$$= \frac{L^3}{2} - \frac{L^3}{3} = \frac{3-1}{6}L^3 = \frac{1}{6}L^3$$

(4) Obtain the mass matrix of a two noded element with variable x-sectional area (linearly varying from A_1 to A_2

$$A(x) = A_1 N_1(x) + A_2 N_2(x) \qquad (1)$$

the mass matrix will be given by

$$m = \frac{\rho}{L^2}\int_0^L \begin{bmatrix} (L-x)^2 & x(L-x) \\ x(L-x) & x^2 \end{bmatrix} A(x)\, dx \qquad (2)$$

$$= \frac{\rho}{L^2}\left[A_1 \int_0^L \begin{bmatrix} (L-x)^2 & x(L-x) \\ x(L-x) & x^2 \end{bmatrix}(L-x)\, dx\right.$$

$$\left. + A_2 \int_0^1 \begin{bmatrix} (L-x)^2 & x(L-x) \\ x(L-x) & x^2 \end{bmatrix} x\, dx\right]$$

$$= \frac{\rho}{L^2}\left[A_1 \begin{bmatrix} L^3/3 & \frac{L^3}{12} \\ L^3/12 & L^3/3 \end{bmatrix} + A_2 \begin{bmatrix} L^3/12 & L^3/12 \\ L^3/12 & L^3/3 \end{bmatrix}\right]$$

$$= \frac{\rho L}{12}\begin{bmatrix} 3A_1 + A_2 & A_1 + A_2 \\ A_1 + A_2 & A_1 + 3A_2 \end{bmatrix}$$

in the case when $A_1 = A_2$ we obtain the mass matrix of exercise 3

5 A two noded bar element in 3D has only translational D.O.F. Find the diagonal mass matrix of the element.

Solution:

the bar element will have 6 translational D.O.Fs (3 per node in each direction, two nodes) the mass matrix will only have diagonal terms because the D.O.F.s are only translational. therefor

$$m = \frac{\rho A L}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & \text{Symm.} & & & 1 & 0 \\ & & & & & 1 \end{bmatrix}$$