Assignment 1.1

On "The Direct Stiffness Method"

Consider the truss problem defined in the figure 1.1. All geometric and material properties: L, α, E and A, as well as the applied forces P and H are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed-displacement conditions at nodes 2, 3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.

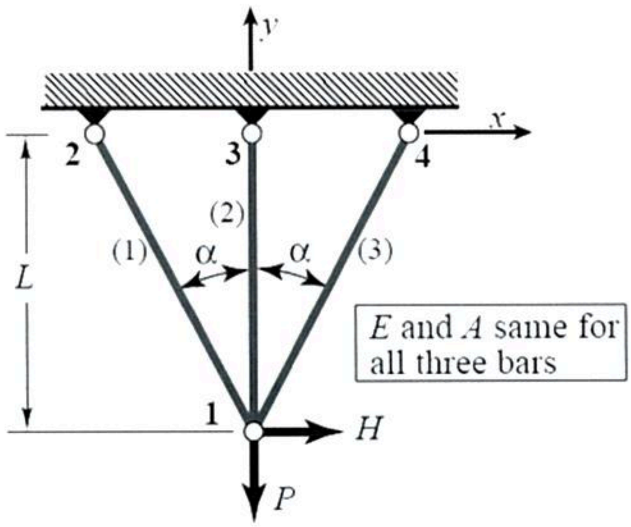


Figure 1.1.- Truss structure. Geometry and mechanical features

1. Show that the master stiffness equations are.

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{symm} & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

in which c = cosα and s = sinα. Explain from physics why the 5^{th} row and column contain only zeros.

2. Apply the BC's and show the 2-equation modified stiffness system.

3. Solve for the displacements u_{x1} and u_{y1}. Check that the solution makes physical sense for the limit cases $\alpha \to 0$ and $\alpha \to \pi/2$. Why does u_{x1} "blow up" if $H \neq 0$ and $\alpha \to 0$?

4. Recover the axial forces in the three members. Partial answer: $F^{(3)} = -H/(2s) + Pc^2/(1+2c^3)$. Why do $F^{(1)}$ and $F^{(3)}$ "blow up" if $H \neq 0$ and $\alpha \to 0$?

5. Dr. Who proposes "improving" the result for the example truss of the 1^{st} lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr. Who's suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.

Assignment 1.2

Dr. Who proposes "improving" the result for the example truss of the 1^{st} lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr. Who's suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.

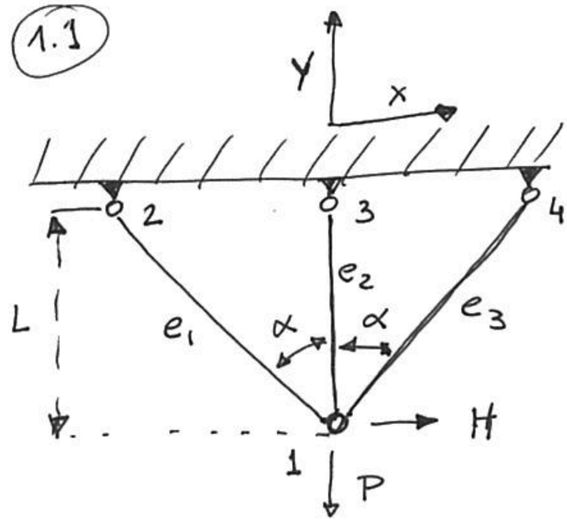
Date of Assignment: **5 / 02 / 2018**
Date of Submission: **12 / 02 / 2018**

The assignment must be submitted as a pdf file named **As1-Surname.pdf** to the CIMNE virtual center.

① CSMD Assignment 1

(1.1)

José Andino Saint Antoni



H, P : applied forces

α, L, A: geometry of system

E : Young's modulus.

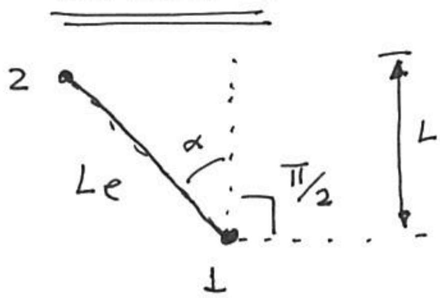
(1.1.1) Find the master stiffness equations

We need to write the stiffness matrix for each element and then assemble the global matrix.

taking origin in node 3 the nodal positions are:

node	x	Y
1	0	$-L$
2	$-L\sin\alpha$	0
3	0	0
4	$L\sin\alpha$	0

Element 1



$L_e \cos\alpha = L$

$L_e = \frac{L}{\cos\alpha} = \frac{L}{c}$

$\sin\varphi = \sin(\alpha + \frac{\pi}{2})$
$= \cos\alpha = c$

$\cos\varphi = \cos(\alpha + \frac{\pi}{2}$
$= -\sin\alpha = -$

Using the element stiffness matrix

(2)

$$K_e = \frac{E^e A^e}{L^e} \begin{bmatrix} \cos^2\varphi & \sin\varphi\cos\varphi & -\cos^2\varphi & -\sin\varphi\cos\varphi \\ \vdots & \sin^2\varphi & -\cos\varphi\sin\varphi & -\sin^2\varphi \\ \vdots & \vdots & \cos^2\varphi & \sin\varphi\cos\varphi \\ \text{symmetric} \ldots & & \ldots & \sin^2\varphi \end{bmatrix}$$

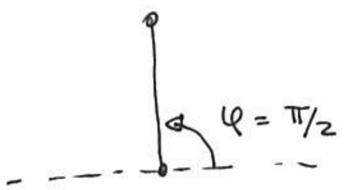
we set, using $\sin\varphi = c$ and $\cos\varphi = -s$

$$K_1 = \frac{EA\,c}{L} \begin{bmatrix} s^2 & -sc & -s^2 & sc \\ & c^2 & sc & -c^2 \\ \vdots & & s^2 & -sc \\ \text{sym} \ldots & & & c^2 \end{bmatrix}$$

For element 2

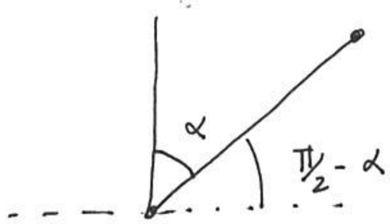
$L_e = L \qquad \cos\varphi = \cos \pi/2 = 0$

$\qquad\qquad\quad \sin\varphi = \sin \pi/2 = 1$



$$K_2 = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ \vdots & & 0 & 0 \\ \text{sym.} \ldots & & & 1 \end{bmatrix}$$

Element 3



$L_e = \frac{L}{c}$

$\sin\varphi = \sin\left(\frac{\pi}{2} - \alpha\right)$

$= \cos\alpha = c$

$\cos\varphi = \cos\left(\frac{\pi}{2} - \alpha\right)$

$= \sin\alpha = s$

(3)

$$k_3 = \frac{EA}{L}\begin{bmatrix} cs^2 & sc^2 & -cs^2 & -sc^2 \\ \vdots & c^3 & -sc^2 & -c^3 \\ \vdots & \vdots & cs^2 & sc^2 \\ \text{symm} & \cdots & \cdots & c^3 \end{bmatrix}$$

where we multiplied the matrix elements by c as $Le = L/c$

<u>the force vector are:</u>

$$\bar{F} = \left[F_{x_1} \ F_{y_1} \ F_{x_2} \ F_{y_2} \ F_{x_3} \ F_{y_3} \ F_{x_4} \ F_{y_4} \right]^T$$

$$= \left[\underline{H} \ \ \underline{-P} \ F_{x_2} \ F_{y_2} \ F_{x_3} \ F_{y_3} \ F_{x_4} \ F_{y_4} \right]^T$$

where H and P are known but the reactions F_{x_i} F_{y_i} $i = \{2, 3, 4\}$ are unknowns.

<u>The displacement vector is:</u>

$$\bar{u} = \left[u_{x_1} \ u_{y_1} \ u_{x_2} \ u_{y_2} \ u_{x_3} \ u_{y_3} \ u_{x_4} \ u_{y_4} \right]^T$$

$$\bar{u} = \left[u_{x_1} \ u_{y_1} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \right]^T$$

where only u_{x_1} and u_{y_1} are unknown, the rest are prescribed as being zero.

To assemble the Global stiffness matrix we need to know how local and global nodes relate.

4

	local	Global	mat. Element
e_1	1 x	1 x	1
	y	y	2
	2 x	2 x	3
	y	y	4
e_2	1 x	1	1
	y		2
	2 x	3	5
	y		6
e_3	1 x	1	1
	y		2
	2 x	4	7
	y		8

Assembling the global matrix we get:

$$K = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ \vdots & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ \vdots & \cdots & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ \vdots & & \vdots & c^3 & 0 & 0 & 0 & 0 \\ \vdots & & \vdots & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix}$$

Row and colum 5 contain only zeroes. They are releted to horizontal displacement and force on node 3. Because of the assumption that α is equal on both sides, the bar element 2 can only have vertical forces, there is no possible displacement or force in the x direction for node 3.

5

1.1.2 the border conditions are

$$\mu_{ix} = \mu_{iy} = 0 \qquad i = \{2, 3, 4\}$$

the global matrix reduces to

$$\frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \cdot \begin{bmatrix} \mu_{x1} \\ \mu_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

1.1.3 The system matrix is diagonal, so we can write directly:

$$\begin{cases} \frac{EA}{L} 2cs^2 \mu_{x1} = H \\ \frac{EA}{L} (1+2c^3) \mu_{y1} = -P \end{cases}$$

or explicitly:

$$\begin{cases} \mu_{x1} = \dfrac{HL}{EA\ 2 \cos\alpha \sin^2\alpha} \\ \mu_{y1} = \dfrac{-PL}{EA\ (1+2\cos^3\alpha)} \end{cases}$$

for $\alpha \to 0 \qquad \mu_{x1} \to \infty$

this is because as modes 2 and 4 come closer to mode 3, the system tends to become a pendulum, wich is not statically determined.

(6) also for $\alpha \to 0$ $\quad u_{y1} \to \dfrac{-PL}{EA\,(3)}$

this is the result of applying Hooke's law on three vertical parallel bars $\quad F_y = -ky$

$$\Rightarrow \quad P = -3\,\frac{EA}{L}\,u_{y1}$$

the deformation is linear elastic on the vertical axis.

(1.1.4) To recover the nodal forces all we need to do is multiply $\bar{K}\,\bar{u}$ as $\bar{u}$ is now known

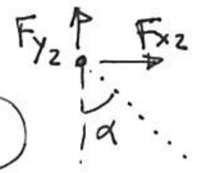
$$\bar{u} = \left[\frac{HL}{EA\,2cs^2}\,,\ \frac{-PL}{EA\,(1+2c^3)}\,,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right]^T$$

only the first two columns are multiplied by non-zero elements, so we can ignore the rest. also, the first two rows are equal to H and $-P$ so we can ignore too.

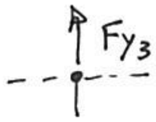
(7) what is left of $\bar{K}\bar{u} = \bar{f}$ is

$$\begin{bmatrix} F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix} = \begin{bmatrix} -cs^2 & c^2 s \\ c^2 s & -c^3 \\ 0 & 0 \\ 0 & -1 \\ -cs^2 & -c^2 s \\ -c^2 s & -c^3 \end{bmatrix} \cdot \begin{bmatrix} \dfrac{H}{2cs^2} \\ \dfrac{-P}{(1+2c^3)} \end{bmatrix} \frac{L}{EA}$$

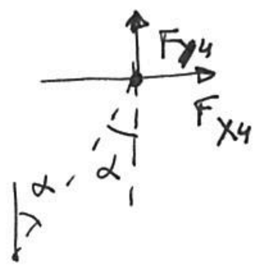
for each mode we can recover the axial force:

(e1) 

$$f_1 = \frac{F_{y2}}{\cos\alpha} = \frac{H}{2s} - \frac{Pc^2}{1+2c^3}$$

(e2) 

$$f_2 = F_{y3} = \frac{P}{1+2c^3}$$

(e3) 

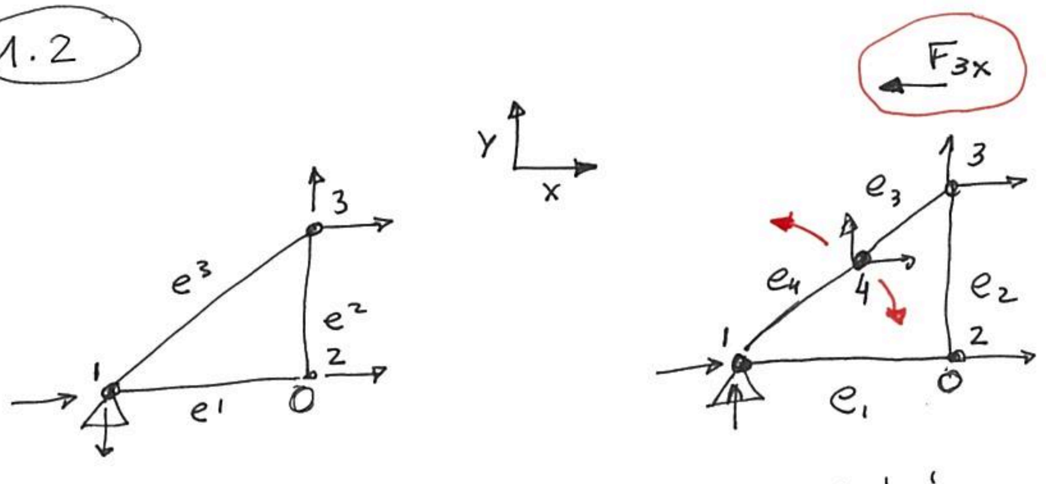
$$f_3 = \frac{F_{y4}}{\cos\alpha} = -\frac{H}{2s} + \frac{Pc^2}{1+2c^3}$$

as $\alpha \to 0$ with $H \neq 0$, f_1 and f_3 tend to infinity as even a small horizontal force would require a large stress to prevent the rotation around 3.

for $\alpha = 0$, modes 3 and 4 overlap and the system is no longer statically determin

8

1.2

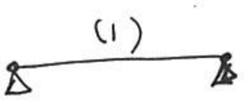


original Problem. Dr Who's 'improved' version

Intuitively, Dr Who's model is undetermined as the node 4 can move freely up or down if there is a negative force on node 3 (indicated in red)

Using the general stiffness matrix for each element:

<u>Element 1</u>



$\varphi = 0$

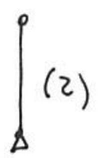
$\frac{EA}{L} = 10$

$\cos 0 = 1$

$\sin 0 = 0$

$$K_1 = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ \vdots & 0 & 0 & 0 \\ & \vdots & 1 & 0 \\ \text{sym} & & \cdots & 0 \end{bmatrix}$$

<u>Element 2</u>



$\varphi = \pi/2$

$\frac{EA}{L} = 5$

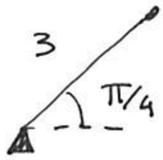
$\cos \pi/2 = 0$

$\sin \pi/2 = 1$

$$K_2 = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & 1 & 0 & 0 \\ \vdots & \vdots & 1 & 0 \\ \text{sym.} & & \cdots & 0 \end{bmatrix}$$

9

Element 3 (Dr Who's problem)



$$\frac{EA}{L} = 40$$

$$\sin \pi/4 = \cos \pi/4 = \frac{\sqrt{2}}{2}$$

$$K_3 = 20 \begin{bmatrix} 1 & 1 & -1 & -1 \\ \cdot & 1 & -1 & -1 \\ \cdot & \cdot & 1 & 1 \\ \text{sym} & \cdot & \cdot & 1 \end{bmatrix}$$

Element 4 (Dr Who's problem)

identical to e^3

Displacement vector

$$\bar{u} = [\underline{0 \;\; 0} \;\; u_{x2} \;\; \underline{0} \;\; u_{x3} \;\; u_{y3} \;\; u_{x4} \;\; u_{y4}]^T$$

Forces vector

$$\bar{F} = [F_{x1} \;\; F_{y1} \;\; F_{x2} \;\; F_{y2} \;\; \underline{2 \;\; 1} \;\; F_{x4} \;\; F_{y4}]$$

in both cases we have applied the border conditions given (in red)

the global system is $\bar{K}\bar{u} = \bar{f}$

$$\begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ \cdot & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ \cdot & \cdot & 10 & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & 5 & 0 & -5 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 20 & 20 & -20 & -20 \\ \cdot & & & & \cdot & 25 & -20 & -20 \\ \cdot & \text{symm.} & & & & \cdot & 40 & 40 \\ \cdot & & & & & & \cdot & 40 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_{x2} \\ 0 \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ 2 \\ 1 \\ F_{x4} \\ F_{y4} \end{bmatrix}$$

(10) Introducing the matrix into Matlab we get $\det(\bar{K}) = 0$ this means that the row vectors are not linearly independent and the matrix is singular therefore, there is no unique solution This translates into multiple possible solutions. The additional DOF introduced makes the structure undetermined.