Assignment 1

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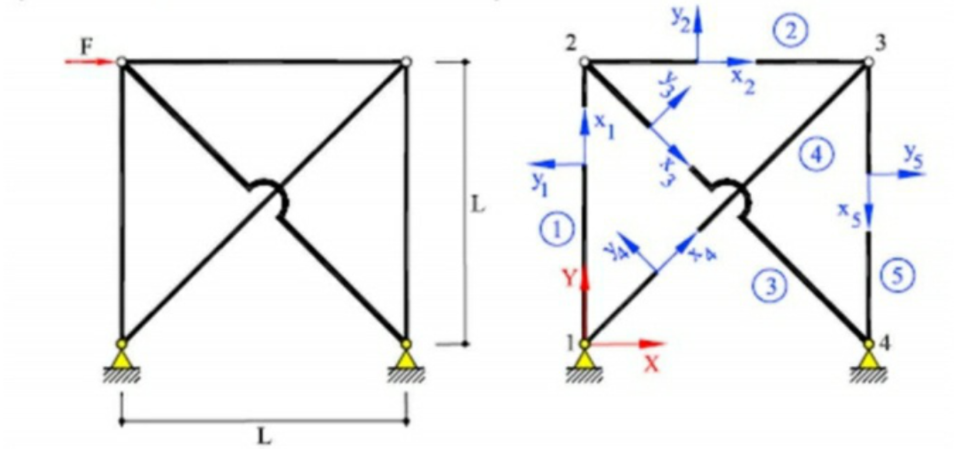
Master in Numerical Methods in Engineering

Assignment 1.1

Solve the hyperstatic structure shown in the figure subjected to the indicated loads. The axial stiffness EA of the bar is equal for all bars.

$L = 6$ m, $A = 6$ cm^2, $E = 200$ GPa, $F = 80$ kN



$$k^e = \frac{EA}{L}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \quad \text{with } c = \cos\theta, \; s = \sin\theta$$

For elements 1, 2 and 5 $\rightarrow$ $\frac{EA}{L} = \frac{(200\times10^9)(0{,}006)}{6} = 2\times10^7$

For elements 3 and 4 $\rightarrow$ $\frac{EA}{L} = \frac{(200\times10^9)(0{,}006)}{6\sqrt{2}} = 1{,}414\times10^7$

So for element 1 $(\theta = \pi/2)$

$$\begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{bmatrix} = 2\times10^7 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix}$$

For element 2 $(\theta = 0)$

$$\begin{bmatrix} F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \end{bmatrix} = 2\times10^{7} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

For element 3 $(\theta = -\pi/4)$

$$\begin{bmatrix} F_{x2} \\ F_{y2} \\ F_{x4} \\ F_{y4} \end{bmatrix} = 1,414\times10^{7} \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

For element 4 $(\theta = \pi/4)$

$$\begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x3} \\ F_{y3} \end{bmatrix} = 1,414\times10^{7} \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

For element 5 $(\theta = -\pi/2)$

$$\begin{bmatrix} F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix} = 2\times10^{7} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Applying the equilibrium rule

$$f = \sum f_i = f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)} + f^{(5)}$$

and forming the master stiffness equations through equilibrium rule

$$f = f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)} + f^{(5)} = (K^{(1)} + K^{(2)} + K^{(3)} + K^{(4)} + K^{(5)})u = Ku$$

$$\begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{Bmatrix} = \begin{bmatrix} B & B & 0 & 0 & -B & -B & 0 & 0 \\ B & A+B & 0 & -A & -B & -B & 0 & 0 \\ 0 & 0 & A+B & -B & -A & 0 & -B & B \\ 0 & -A & -B & A+B & 0 & 0 & B & -B \\ -B & -B & -A & 0 & A+B & B & 0 & 0 \\ -B & -B & 0 & 0 & B & A+B & 0 & -A \\ 0 & 0 & -B & B & 0 & 0 & B & -B \\ 0 & 0 & B & -B & 0 & -A & -B & B+A \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix}$$

with $A = 2\times10^7$ $\quad B = 1.141\times10^7/2 = 7.071\times10^7$

Applying support and loading boundary conditions

- $u_{x1} = u_{y1} = u_{x4} = u_{y4} = 0$ → Displacements in the supports
- $F_{x2} = 80000$ N, $F_{y2} = F_{x3} = F_{y3} = 0$ → Forces in the upper nodes

thus

$$\begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{Bmatrix} = \begin{bmatrix} B & B & 0 & 0 & -B & -B & 0 & 0 \\ B & A+B & 0 & -A & -B & -B & 0 & 0 \\ 0 & 0 & A+B & -B & -A & 0 & -B & B \\ 0 & -A & -B & A+B & 0 & 0 & B & -B \\ -B & -B & -A & 0 & A+B & B & 0 & 0 \\ -B & -B & 0 & 0 & B & A+B & 0 & -A \\ 0 & 0 & -B & B & 0 & 0 & B & -B \\ 0 & 0 & B & -B & 0 & -A & -B & B+A \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix}$$

Separating the equations 3, 4, 5 and 6 we obtain

- $80000 = (A+B)\,U_{x2} - B\,U_{y2} - A\,U_{x3}$
- $0 = -B\,U_{x2} + (A+B)\,U_{y2}$
- $0 = -A\,U_{x2} + (A+B)\,U_{x3} + B\,U_{y3}$
- $0 = B\,U_{x3} + (A+B)\,U_{y3}$

Solving we obtain

$U_{x2} \simeq 0{,}0085413$ m $\qquad U_{y2} \simeq 0{,}0067724$ m

$U_{x3} \simeq 0{,}0022310$ m $\qquad U_{y3} \simeq -0{,}0017690$ m //

With these values we can calculate the reactions on node ① and ④, so:

$f_{x1} = -B\,U_{x3} - B\,U_{y3} \simeq -35379.38$ N //

$f_{y1} = -A\,U_{y2} - B\,U_{x3} - B\,U_{y3} \simeq -80000$ N //

$f_{x4} = -B\,U_{x2} + B\,U_{y2} \simeq -44620{,}55$ N //

$f_{y4} = B\,U_{x2} - B\,U_{y2} - A\,U_{y3} \simeq 80000$ N //

Assignment 1.2

Dr. Who proposes "improving" the result for the example truss of the 1st lesson by putting one extra node, 5 at the midpoint of member (4) 1-3, so that it is subdivided in two different members: (3) 1-5 and (4) 3-5. His "reasoning" is that more is better. Try Dr. Who's suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.

The stiffness matrix of the elements 1, 2, 4 and 5 remain. So, the stiffness matrix of the elements 1-5 and 3-5 are: ($\theta = \pi/4$ for both elements)

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x5} \\ f_{y5} \end{bmatrix} = B \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x5} \\ u_{y5} \end{bmatrix} \rightarrow \text{Element } 1-5$$

$$\begin{bmatrix} f_{x3} \\ f_{y3} \\ f_{x5} \\ f_{y5} \end{bmatrix} = B \begin{array}{c} \\ 5 \\ \\ 3 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} \quad & 5 & \quad & 3 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} \end{array} \begin{bmatrix} u_{x3} \\ u_{y3} \\ u_{x5} \\ u_{y5} \end{bmatrix} \rightarrow \text{Element } 5-3$$

Assembling the master stiffness matrix we obtain

$$K = \begin{bmatrix} B & B & 0 & 0 & 0 & 0 & 0 & 0 & -B & -B \\ B & A+B & 0 & -A & 0 & 0 & 0 & 0 & -B & -B \\ 0 & 0 & A+B & -B & -A & 0 & -B & B & 0 & 0 \\ 0 & -A & -B & A+B & 0 & 0 & B & -B & 0 & 0 \\ 0 & 0 & -A & 0 & A+B & B & 0 & 0 & -B & -B \\ 0 & 0 & 0 & 0 & B & A+B & 0 & -A & -B & -B \\ 0 & 0 & -B & B & 0 & 0 & B & -B & 0 & 0 \\ 0 & 0 & B & -B & 0 & -A & -B & A+B & 0 & 0 \\ -B & -B & 0 & 0 & -B & -B & 0 & 0 & B & B \\ -B & -B & 0 & 0 & -B & -B & 0 & 0 & B & B \end{bmatrix}$$

The system can not be solved due to the equations given by the node # 5 are linearly dependent. Physically, this due to this node doesn't have any restriction.

Assignment 1.3

On "The Direct Stiffness Method"

Consider the truss problem defined in the figure 1.1. All geometric and material properties: L, α, E and A, as well as the applied forces P and H are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed-displacement conditions at nodes 2, 3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.

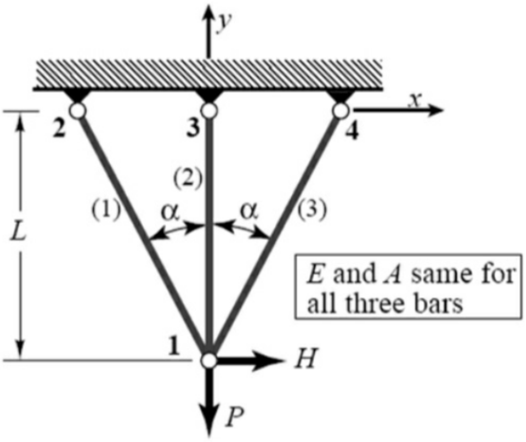


Figure 1.1.- Truss structure. Geometry and mechanical features

1. Show that the master stiffness equations are,

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{symm} & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

in which c = cosα and s = sinα. Explain from physics why the 5th row and column contain only zeros.

2. Apply the BC's and show the 2-equation modified stiffness system.

3. Solve for the displacements u_{x1} and u_{y1}. Check that the solution makes physical sense for the limit cases $\alpha \to 0$ and $\alpha \to \pi/2$. Why does u_{x1} "blow up" if $H \neq 0$ and $\alpha \to 0$?

4. Recover the axial forces in the three members. Partial answer: $F^{(3)} = -H/(2s) + Pc^2/(1+2c^3)$. Why do $F^{(1)}$ and $F^{(3)}$ "blow up" if $H \neq 0$ and $\alpha \to 0$?

a)

$\theta_1 = 90° + \alpha \qquad \theta_2 = 90° \qquad \theta_3 = 90° - \alpha$

Having that $\cos\alpha = c$ and $\sin \alpha = s$ we can say

$\sin\theta_1 = c \qquad \cos\theta_1 = -s$

$\text{Sen}\,\theta_2 = 1 \qquad \cos\theta_2 = 0$

$\text{Sen}\,\theta_3 = c \qquad \cos\theta_3 = s$

For element 1

$$k^{(1)} = \left(\frac{EA}{(L/c)}\right)^{(1)} \begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix} = \left(\frac{EA}{L}\right)^{(1)} \begin{bmatrix} cs^2 & -c^2 s & -cs^2 & c^2 s \\ -c^2 s & c^3 & c^2 s & -c^3 \\ -cs^2 & c^2 s & cs^2 & -c^2 s \\ c^2 s & -c^3 & -c^2 s & c^3 \end{bmatrix}$$

$$k^{(2)} = \left(\frac{EA}{L}\right)^{(2)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$k^{(3)} = \left(\frac{EA}{L}\right)^{(3)} \begin{bmatrix} cs^2 & c^2 s & -cs^2 & -c^2 s \\ c^2 s & c^3 & -c^2 s & -c^3 \\ -cs^2 & -c^2 s & cs^2 & c^2 s \\ -c^2 s & -c^3 & c^2 s & c^3 \end{bmatrix}$$

Assembling the global matrix we obtain

$$K = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2 s & 0 & 0 & -cs^2 & -c^2 s \\ & 1+2c^3 & c^2 s & -c^3 & 0 & -1 & -c^2 s & -c^3 \\ & & cs^2 & -c^2 s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & \text{Sym.} & & & & & cs^2 & c^2 s \\ & & & & & & & c^3 \end{bmatrix}$$

And the displacements vector has the form

$$u = \begin{bmatrix} u_{x1} & u_{y1} & u_{x2} & u_{y2} & u_{x3} & u_{y3} & u_{x4} & u_{y4} \end{bmatrix}^T$$

And the force vector has the form

$$f = [H \quad -P \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

And the system has the form $Ku = f$ //

the element 2 is completely vertical and the scheme is symmetric on y-axis, for this reason the displacements on x-axis of node # 3 don't have any effect on our system.

b) the nodes 2, 3 and 4 are fixed, therefore the displacements on these nodes will be zero.

$$u = [u_{x_1} \quad u_{y_1} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

this take us to solve the first two equations of our system, so:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x_1} \\ u_{y_1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

c) Solving the system we obtain

$$u_{x_1} = \frac{HL}{2EA\,cs^2} \quad // \qquad u_{y_1} = \frac{-PL}{EA}\left(\frac{1}{1+2c^3}\right) \quad //$$

when $\alpha = 0$	when $\alpha = \pi/2$
$u_{x_1} \to \infty$	$u_{x_1} \to \infty$
$u_{y_1} \to -\frac{PL}{3EA}$	$u_{y_1} \to -\frac{PL}{EA}$

From the answers above, if $H \neq 0$ and $\alpha \to 0$ the answer "blows-up"

d) $\bar{u}^e = T^e u^e$

$$\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix}$$

Element ①

$$\begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x2} \\ \bar{u}_{y2} \end{bmatrix} = \begin{bmatrix} -HL/2EAcs - PLc/EA(1+2c^3) \\ -HL/2EAs^2 - PLs/EA(1+2c^3) \\ 0 \\ 0 \end{bmatrix}$$

Elongation $d = -\bar{u}_{x1} = \dfrac{HL}{2EAcs} + \dfrac{PLc}{EA(1+2c^3)}$

Axial force $F = \dfrac{EA}{Lc} d = \dfrac{H}{2s} + \dfrac{Pc^2}{1+2c^3}$

Element ②

$$\begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x3} \\ \bar{u}_{y3} \end{bmatrix} = \begin{bmatrix} -PL/EA(1+2c^3) \\ -HL/2EAcs^2 \\ 0 \\ 0 \end{bmatrix}$$

Elongation $d = -\bar{u}_{x1} = \dfrac{PL}{EA(1+2c^3)}$

Axial force $F = \dfrac{EA}{L} d = \dfrac{P}{1+2c^3}$

Element ③

$$\begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x4} \\ \bar{u}_{y4} \end{bmatrix} = \begin{bmatrix} HL/2EAcs - PLc/EA(1+2c^3) \\ -HL/2EAs^2 - PLs/EA(1+2c^3) \\ 0 \\ 0 \end{bmatrix}$$

Elongation $d = -\bar{u}_{x1} = \dfrac{-HL}{2EA\,cs} - \dfrac{PLc}{EA(1+2c^3)}$

Axial force $F = \dfrac{EA}{L/c} d = -\dfrac{H}{2s} + \dfrac{Pc^2}{1+2c^3}$

For $H \neq 0$ and $\alpha \to 0 \to \dfrac{H}{2s} \to \infty$ //