Universitat Politècnica de Catalunya
Master of Science in Computational Mechanics
Computational Strutural Mechanics and Dynamics
CSMD
Spring Semester 2017/2018

Homework 1 -The Direct Stiffness Method

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February 12, 2018

Assingment 1

a)

The general form of the elemental stiffness matrix is given by:

$$\mathbf{K^{(e)}} = \left(\frac{EA}{L}\right)^e \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \tag{1}$$

Where $c = cos(\varphi)$ and $s = sin(\varphi)$, and φ is the angle between the horizontal axis of the global coordinate system and the element axial line.

Evaluation of elemental stiffness matrices for the structure:

Element 1:

For element 1 we have:
$\varphi = \frac{\pi}{2} + \alpha$, thus: $c = cos\varphi = cos(\frac{\pi}{2} + \alpha) = -sin\alpha$ and $s = sin\varphi = sin(\frac{\pi}{2} + \alpha) = cos\alpha$. Further $L^{(1)} = \frac{L}{cos\alpha}$
From this result, equation (1), for element 1, can be written as:

$$\mathbf{K^{(1)}} = cos(\alpha)\left(\frac{EA}{L}\right) \begin{bmatrix} sin(\alpha)^2 & -cos(\alpha)sin(\alpha) & -sin(\alpha)^2 & cos(\alpha)sin(\alpha) \\ & cos(\alpha)^2 & cos(\alpha)sin(\alpha) & -cos(\alpha)^2 \\ & & sin(\alpha)^2 & -cos(\alpha)sin(\alpha) \\ \text{Symm.} & & & cos(\alpha)^2 \end{bmatrix} \tag{2}$$

Writing $sin\alpha = s$ and $cos\alpha = c$, equation (2) becomes:

$$\mathbf{K^{(1)}} = \left(\frac{EA}{L}\right) \begin{bmatrix} cs^2 & -c^2 s & -cs^2 & c^2 s \\ & c^3 & c^2 s & -c^3 \\ & & cs^2 & -c^2 s \\ \text{Symm.} & & & c^3 \end{bmatrix} \tag{3}$$

Element 2:

For element 2 we have:
$\varphi = \frac{\pi}{2}$, thus: $c = cos\varphi = cos(\frac{\pi}{2}) = 0$ and $s = sin\varphi = sin(\frac{\pi}{2}) = 1$. Further $L^{(2)} = L$
From this result, equation (1), for element 2, can be written as:

$$\mathbf{K^{(2)}} = \left(\frac{EA}{L}\right) \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ \text{Symm.} & & & 1 \end{bmatrix} \tag{4}$$

Element 3:

For element 3 we have:
$\varphi = \frac{\pi}{2} - \alpha$, thus: $c = cos\varphi = cos(\frac{\pi}{2} - \alpha) = sin\alpha$ and $s = sin\varphi = sin(\frac{\pi}{2} - \alpha) = cos\alpha$. Further $L^{(3)} = \frac{L}{cos\alpha}$
From this result, equation (1), for element 3, can be written as:

$$\mathbf{K^{(3)}} = cos(\alpha) \left(\frac{EA}{L}\right) \begin{bmatrix} sin(\alpha)^2 & cos(\alpha)sin(\alpha) & -sin(\alpha)^2 & -cos(\alpha)sin(\alpha) \\ & cos(\alpha)^2 & -cos(\alpha)sin(\alpha) & -cos(\alpha)^2 \\ & & sin(\alpha)^2 & cos(\alpha)sin(\alpha) \\ \text{Symm.} & & & cos(\alpha)^2 \end{bmatrix} \tag{5}$$

Writing $sin\alpha = s$ and $cos\alpha = c$, equation (5) becomes:

$$\mathbf{K^{(3)}} = \left(\frac{EA}{L}\right) \begin{bmatrix} cs^2 & c^2s & -cs^2 & -c^2s \\ & c^3 & -c^2s & -c^3 \\ & & cs^2 & c^2s \\ \text{Symm.} & & & c^3 \end{bmatrix} \tag{6}$$

Assembling global stiffness matrix

The expanded stiffness equations for element 1 are:

$$\begin{Bmatrix} f_{x_1}^{(1)} \\ f_{y_1}^{(1)} \\ f_{x_2}^{(1)} \\ f_{y_2}^{(1)} \\ f_{x_3}^{(1)} \\ f_{y_3}^{(1)} \\ f_{x_4}^{(1)} \\ f_{y_4}^{(1)} \end{Bmatrix} = \left(\frac{EA}{L}\right) \begin{bmatrix} cs^2 & -c^2s & -cs^2 & c^2s & 0 & 0 & 0 & 0 \\ & c^3 & c^2s & -c^3 & 0 & 0 & 0 & 0 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ \text{Symm.} & & & & & & & 0 \end{bmatrix} \begin{Bmatrix} u_{x_1}^{(1)} \\ u_{y_1}^{(1)} \\ u_{x_2}^{(1)} \\ u_{y_2}^{(1)} \\ u_{x_3}^{(1)} \\ u_{y_3}^{(1)} \\ u_{x_4}^{(1)} \\ u_{y_4}^{(1)} \end{Bmatrix} \tag{7}$$

The expanded stiffness equations for element 2 are:

$$\begin{Bmatrix} f_{x_1}^{(2)} \\ f_{y_1}^{(2)} \\ f_{x_2}^{(2)} \\ f_{y_2}^{(2)} \\ f_{x_3}^{(2)} \\ f_{y_3}^{(2)} \\ f_{x_4}^{(2)} \\ f_{y_4}^{(2)} \end{Bmatrix} = \left(\frac{EA}{L}\right) \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & 0 & 0 \\ \text{Symm.} & & & & & & & 0 \end{bmatrix} \begin{Bmatrix} u_{x_1}^{(2)} \\ u_{y_1}^{(2)} \\ u_{x_2}^{(2)} \\ u_{y_2}^{(2)} \\ u_{x_3}^{(2)} \\ u_{y_3}^{(2)} \\ u_{x_4}^{(2)} \\ u_{y_4}^{(2)} \end{Bmatrix} \tag{8}$$

The expanded stiffness equations for element 3 are:

$$\begin{Bmatrix} f_{x_1}^{(3)} \\ f_{y_1}^{(3)} \\ f_{x_2}^{(3)} \\ f_{y_2}^{(3)} \\ f_{x_3}^{(3)} \\ f_{y_3}^{(3)} \\ f_{x_4}^{(3)} \\ f_{y_4}^{(3)} \end{Bmatrix} = \left(\frac{EA}{L}\right) \begin{bmatrix} cs^2 & c^2s & 0 & 0 & 0 & 0 & -cs^2 & -c^2s \\ & c^3 & 0 & 0 & 0 & 0 & -c^2s & -c^3 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{Symm.} & & & & & & & c^3 \end{bmatrix} \begin{Bmatrix} u_{x_1}^{(3)} \\ u_{y_1}^{(3)} \\ u_{x_2}^{(3)} \\ u_{y_2}^{(3)} \\ u_{x_3}^{(3)} \\ u_{y_3}^{(3)} \\ u_{x_4}^{(3)} \\ u_{y_4}^{(3)} \end{Bmatrix} \tag{9}$$

From the application of compatibility on displacements and equilibrium of forces in each nodes, the upper indexes can be dropped and the stiffness matrices on equations (7), (8) and (9) can be summed-up leading to the following:

$$\begin{Bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{Bmatrix} = \left(\frac{EA}{L}\right) \begin{bmatrix} (cs^2+cs^2) & (-c^2s+c^2s) & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & (c^3+1+c^3) & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^3 & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{Symm.} & & & & & & & c^3 \end{bmatrix} \begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{Bmatrix} \tag{10}$$

Thus, the global (master) stiffness matrix ($\mathbf{K}$) is given by:

$$\mathbf{K} = \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^3 & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{Symm.} & & & & & & & c^3 \end{bmatrix} \tag{11}$$

Given the orientation of element 2 (along global y direction), and the fact that node 3 is not being shared with any other element,no forces in x direction at this node ($fx_3 = 0$) can exist, as expected for an axial loaded element. Thus, the whole 5^{th} row must be zero to satisfy. Its worth to recall that the K_5j component of the stiffness matrix relates the force and the 5^{th} D.O.F. (u_{x_3} in this case) with the displacement of the j^{th} D.O.F.

b)

Appling the following set of B.C. on forces and displacements: $f_{x_1} = H$; $f_{y_1} = -P$; $u_{x_i} = u_{y_i} = 0$ for $i = 2, 3, 4.$, the reduced (modified) stiffness system becomes:

$$\left(\frac{EA}{L}\right) \begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \begin{Bmatrix} u_{x_1} \\ u_{y_1} \end{Bmatrix} = \begin{Bmatrix} H \\ -P \end{Bmatrix} \tag{12}$$

c)

From equation (12) we obtain the displacements:

$$u_{x_1} = \frac{HL}{2EA} \frac{1}{cs^2} \tag{13}$$

$$u_{y_1} = \frac{-PL}{EA}\frac{1}{1+2c^3} \tag{14}$$

Limit case: $\alpha \rightarrow 0$

$$\lim_{\alpha\rightarrow 0} u_{x_1} = \frac{HL}{2EA}\lim_{\alpha\rightarrow 0}\frac{1}{cs^2} = \infty$$

$$\lim_{\alpha\rightarrow 0} u_{y_1} = \frac{-PL}{EA}\lim_{\alpha\rightarrow 0}\frac{1}{1+2c^3} = \frac{-PL}{3EA}$$

When $\alpha \rightarrow 0$ for $H \neq 0$, $u_{x_1} \rightarrow \infty$. This is explained by the fact that the modified stiffness matrix (equation (12)) becomes singular. In a physical point of view, there will not be any element resisting loads in x direction, thus force in x direction and the momentum equilibrium can not be satisfied and the structure becomes a mechanism. The result for u_{y_1} is expected, as the 3 bars will be aligned each resisting an axial of load -P/3 in global y direction.

Limit case: $\alpha \rightarrow \pi/2$

$$\lim_{\alpha\rightarrow \pi/2} u_{x_1} = \frac{HL}{2EA}\lim_{\alpha\rightarrow 0}\frac{1}{cs^2} = \infty$$

$$\lim_{\alpha\rightarrow \pi/2} u_{y_1} = \frac{-PL}{EA}\lim_{\alpha\rightarrow 0}\frac{1}{1+2c^3} = \frac{-PL}{EA}$$

The infinity value for u_{x_1} found in this case is only due to the length (L^e) of elements 2 and 3 that will become infinity due to the way they were defined ($L/cos\alpha$). If they are kept finite (e.g., Keeping their lenght constant while α is changed) this singularity is vanished. The result for u_{y_1} is expected as now only one element will be resisting the axial load -P in global y direction.

d)

The general form of the elemental Transformation matrix is given by:

$$\mathbf{T^{(e)}} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \tag{15}$$

Where $c = cos(\varphi)$ and $s = sin(\varphi)$.

Evaluation of elemental axial forces

Element 1

With $cos\varphi = -sin\alpha = -s$ and $sin\varphi = cos\alpha = c$, we obtain.

$$\mathbf{T^{(1)}} = \begin{bmatrix} -sin\alpha & cos\alpha & 0 & 0 \\ -cos\alpha & sin\alpha & 0 & 0 \\ 0 & 0 & -sin\alpha & cos\alpha \\ 0 & 0 & -cos\alpha & -sin\alpha \end{bmatrix} = \begin{bmatrix} -s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix} \tag{16}$$

The displacement in the local axis is given by:

$$\overline{\mathbf{u}}^{(\mathbf{1})} = \mathbf{T^{(1)}}\mathbf{u^{(1)}} = \mathbf{T^{(1)}}\begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -su_{x_1} + cu_{y_1} \\ -cu_{x_1} + su_{y_1} \\ 0 \\ 0 \end{Bmatrix} \tag{17}$$

The element 1 elongation (axial direction) is:

$$d^{(1)} = \overline{u}_{x_2}^{(1)} - \overline{u}_{x_1}^{(1)} = su_{x_1} - cu_{y_1} \tag{18}$$

Thus, from equations (13), (14) and (18) the force axial force in element 1 is given by:

$$F^{(1)} = \frac{EA}{L^{(1)}} d^{(1)} = \frac{EA}{L}(su_{x_1} - cu_{y_1}) = \frac{H}{2s} + \frac{Pc^2}{1+2c^3} \tag{19}$$

Element 2

With $cos\varphi = cos(\pi/2) = 0$ and $sin\varphi = sin(\pi/2) = 1$, we obtain.

$$\mathbf{T^{(2)}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{20}$$

The displacement in the local axis is given by:

$$\overline{\mathbf{u}}^{(2)} = \mathbf{T^{(2)}} \mathbf{u^{(2)}} = \mathbf{T^{(2)}} \begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} u_{y_1} \\ -u_{x_1} \\ 0 \\ 0 \end{Bmatrix} \tag{21}$$

The element 2 elongation (axial direction) is:

$$d^{(2)} = \overline{u}_{x_3}^{(2)} - \overline{u}_{x_1}^{(2)} = -u_{y_1} \tag{22}$$

Thus, from equations (13), (14) and (22) the force axial force in element 1 is given by:

$$F^{(2)} = \frac{EA}{L^{(2)}} d^{(2)} = \frac{EA}{L}(-u_{y_1}) = \frac{P}{1+2c^3} \tag{23}$$

Element 3

With $cos\varphi = sin\alpha = s$ and $sin\varphi = cos\alpha = c$, we obtain.

$$\mathbf{T^{(1)}} = \begin{bmatrix} sin\alpha & cos\alpha & 0 & 0 \\ -cos\alpha & sin\alpha & 0 & 0 \\ 0 & 0 & sin\alpha & cos\alpha \\ 0 & 0 & -cos\alpha & sin\alpha \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \tag{24}$$

The displacement in the local axis is given by:

$$\overline{\mathbf{u}}^{(3)} = \mathbf{T^{(3)}} \mathbf{u^{(3)}} = \mathbf{T^{(3)}} \begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} su_{x_1} + cu_{y_1} \\ -cu_{x_1} + su_{y_1} \\ 0 \\ 0 \end{Bmatrix} \tag{25}$$

The element 3 elongation (axial direction) is:

$$d^{(3)} = \overline{u}_{x_4}^{(3)} - \overline{u}_{x_1}^{(3)} = -su_{x_1} - cu_{y_1} \tag{26}$$

Thus, from equations (13), (14) and (26) the force axial force in element 1 is given by:

$$F^{(3)} = \frac{EA}{L^{(3)}} d^{(3)} = \frac{EA}{L}(-su_{x_1} - cu_{y_1}) = \frac{-H}{2s} + \frac{Pc^2}{1+2c^3} \tag{27}$$

$F^{(1)}$ and $F^{(2)} \to \infty$ when $\alpha \to 0$ because when α increases both axial forces must increase in order to balance the horizontal load H, as those forces projection in x direction is reduced with the increasement of α. For $\alpha = 0$, the limit case where those bars do not exert forces on x direction is reached and the reduced stiffness matrix becomes singular (Mechanism behavior).

Assingment 2

Each element of this "new" truss, proposed by Dr.Who, will have the following physical properties.

Element	Length	EA
1	10	100
2	10	50
3	$5\sqrt{2}$	$200\sqrt{2}$
4	$5\sqrt{2}$	$200\sqrt{2}$

Table 1: Elements physical properties

Evaluation of elemental stiffness matrices for the structure:

Element 1:

For element 1 we have:
$\varphi = 0$, thus: $c = cos\varphi = 1$ and $s = sin\varphi = 0$.From this result, equation (1), for element 1, can be written as:

$$\mathbf{K^{(1)}} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ \text{Symm.} & & & 0 \end{bmatrix} \tag{28}$$

Element 2:

For element 2 we have:
$\varphi = \pi/2$, thus: $c = cos\varphi = 0$ and $s = sin\varphi = 1$.From this result, equation (1), for element 2, can be written as:

$$\mathbf{K^{(2)}} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ \text{Symm.} & & & 1 \end{bmatrix} \tag{29}$$

Elements 3 and 4:

For element 2 we have:
$\varphi = \pi/4$, thus: $c = cos\varphi = \frac{\sqrt{2}}{2}$ and $s = sin\varphi = \frac{\sqrt{2}}{2}$.From this result, equation (1), for elements 3 and 4, can be written as:

$$\mathbf{K^{(3)}} = \mathbf{K^{(4)}} = 20 \begin{bmatrix} 1 & 1 & -1 & -1 \\ & 1 & -1 & -1 \\ & & 1 & 1 \\ \text{Symm.} & & & 1 \end{bmatrix} \tag{30}$$

Assembling global stiffness matrix

The expanded stiffness equations for element 1 are:

$$\begin{Bmatrix} f_{x_1}^{(1)} \\ f_{y_1}^{(1)} \\ f_{x_2}^{(1)} \\ f_{y_2}^{(1)} \\ f_{x_3}^{(1)} \\ f_{y_3}^{(1)} \\ f_{x_4}^{(1)} \\ f_{y_4}^{(1)} \end{Bmatrix} = \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ \text{Symm.} & & & & & & & 0 \end{bmatrix} \begin{Bmatrix} u_{x_1}^{(1)} \\ u_{y_1}^{(1)} \\ u_{x_2}^{(1)} \\ u_{y_2}^{(1)} \\ u_{x_3}^{(1)} \\ u_{y_3}^{(1)} \\ u_{x_4}^{(1)} \\ u_{y_4}^{(1)} \end{Bmatrix} \tag{31}$$

The expanded stiffness equations for element 2 are:

$$\begin{Bmatrix} f_{x_1}^{(2)} \\ f_{y_1}^{(2)} \\ f_{x_2}^{(2)} \\ f_{y_2}^{(2)} \\ f_{x_3}^{(2)} \\ f_{y_3}^{(2)} \\ f_{x_4}^{(2)} \\ f_{y_4}^{(2)} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 5 & 0 & 0 \\ & & & & & & 0 & 0 \\ \text{Symm.} & & & & & & & 0 \end{bmatrix} \begin{Bmatrix} u_{x_1}^{(2)} \\ u_{y_1}^{(2)} \\ u_{x_2}^{(2)} \\ u_{y_2}^{(2)} \\ u_{x_3}^{(2)} \\ u_{y_3}^{(2)} \\ u_{x_4}^{(2)} \\ u_{y_4}^{(2)} \end{Bmatrix} \tag{32}$$

The expanded stiffness equations for element 3 are:

$$\begin{Bmatrix} f_{x_1}^{(3)} \\ f_{y_1}^{(3)} \\ f_{x_2}^{(3)} \\ f_{y_2}^{(3)} \\ f_{x_3}^{(3)} \\ f_{y_3}^{(3)} \\ f_{x_4}^{(3)} \\ f_{y_4}^{(3)} \end{Bmatrix} = \begin{bmatrix} 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & & 20 & 20 \\ \text{Symm.} & & & & & & & 20 \end{bmatrix} \begin{Bmatrix} u_{x_1}^{(3)} \\ u_{y_1}^{(3)} \\ u_{x_2}^{(3)} \\ u_{y_2}^{(3)} \\ u_{x_3}^{(3)} \\ u_{y_3}^{(3)} \\ u_{x_4}^{(3)} \\ u_{y_4}^{(3)} \end{Bmatrix} \tag{33}$$

The expanded stiffness equations for element 4 are:

$$\begin{Bmatrix} f_{x_1}^{(4)} \\ f_{y_1}^{(4)} \\ f_{x_2}^{(4)} \\ f_{y_2}^{(4)} \\ f_{x_3}^{(4)} \\ f_{y_3}^{(4)} \\ f_{x_4}^{(4)} \\ f_{y_4}^{(4)} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & & & 20 & -20 & -20 \\ & & & & & & 20 & 20 \\ \text{Symm.} & & & & & & & 20 \end{bmatrix} \begin{Bmatrix} u_{x_1}^{(4)} \\ u_{y_1}^{(4)} \\ u_{x_2}^{(4)} \\ u_{y_2}^{(4)} \\ u_{x_3}^{(4)} \\ u_{y_3}^{(4)} \\ u_{x_4}^{(4)} \\ u_{y_4}^{(4)} \end{Bmatrix} \tag{34}$$

From the application of compatibility on displacements and equilibrium of forces in each nodes, the upper indexes can be dropped and the stiffness matrices on equations (31), (32),(33) and (34) can be summed-up leading to the following:

$$\begin{Bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{Bmatrix} = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & & & 25 & -20 & -20 \\ & & & & & & 40 & 40 \\ \text{Symm.} & & & & & & & 40 \end{bmatrix} \begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{Bmatrix} \tag{35}$$

Appling the following set of B.C. on forces and displacements: $f_{x_3} = 2$; $f_{y_3} = 1$;$f_{x_2} = f_{x_4} = f_{y_4} = 0$; $u_{x_1} = u_{y_1} = u_{y_2} = 0$, the reduced (modified) stiffness system becomes:

$$\begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & -20 & -20 \\ 0 & 20 & 25 & -20 & -20 \\ 0 & -20 & -20 & 40 & 40 \\ 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{Bmatrix} u_{x_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \tag{36}$$

The modified stiffness matrix($\hat{\mathbf{K}}$) of equation (36) has the two last lines and rows linearly dependent on each other, which leads to a rank of 4 ($rank(\hat{\mathbf{K}}) < dim(\hat{\mathbf{K}})$), thus, $|\hat{\mathbf{K}}| = 0$, and the solution goes to infinity (singular). The two linearly dependent equations are the equations that relates the forces f_{x_4} and f_{y_4} with the displacements in those two elements sharing node 4 (element 3 and element 4). As those two forces are equal (Because 2 equations have same coefficients) the resultant force is acting along those two elements (as expected for axially loaded member). Thus, any external load acting o node 4 having a component in a perpendicular direction to the members orientation (non axial load) would not face any resistance, and under such circumstance the structure is unstable. Adding and extra element sharing node 4 would recover the structure stability,as last two rows and columns will turn linear *independent*, leading to leading to $rank(\hat{\mathbf{K}}) = dim(\hat{\mathbf{K}}) = 5$.However it will reduce the structure degree of determinacy as one more unknown is added to the problem. Other approach could be constrain one of the D.O.F of node 4, thus one of the lines of the modified system of equation can be eliminated leading to $rank(\hat{\mathbf{K}}) = dim(\hat{\mathbf{K}}) = 4$.