Computational Structural Mechanics and Dynamics

Master of Science in Computational Mechanics
Spring Semester 2018

Homework 1

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1987 - 2017 CIMNE 30 years International Centre for Numerical Methods in Engineering

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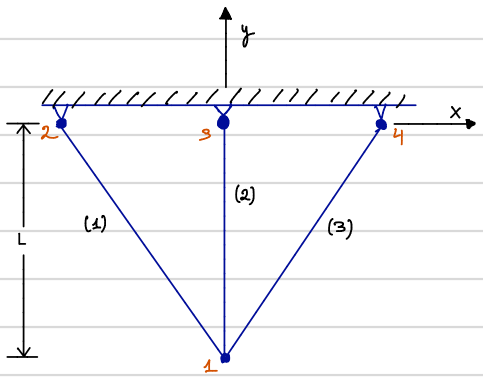
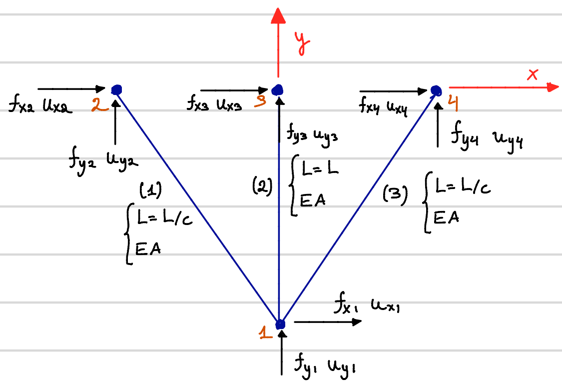
Assignment 1

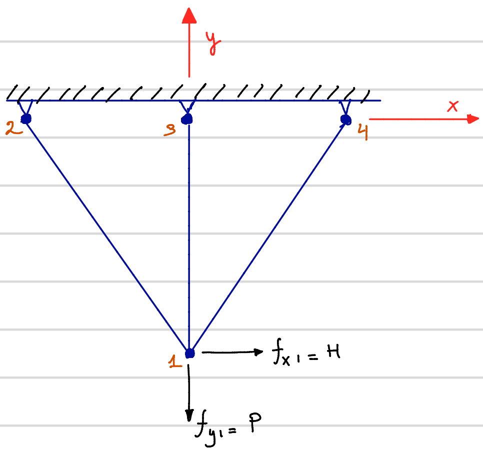
(a) Show that the master stiffness equations are:



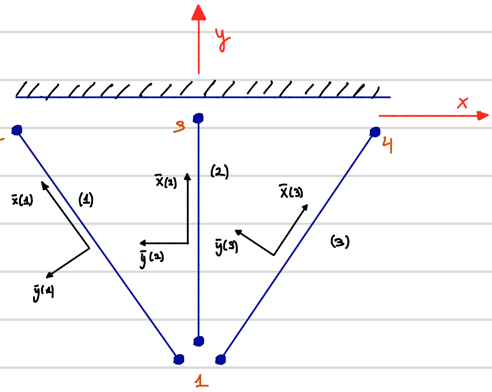
Physical model



Fem Model (8 DoF)

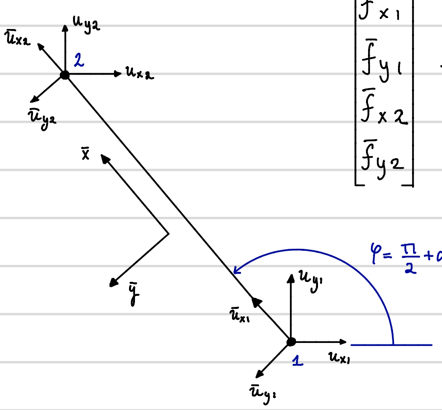


Fem model BCs



Disconnection and Localization

Element (1)



$$\begin{bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1} \\ \bar{f}_{x2} \\ \bar{f}_{y2} \end{bmatrix} = \frac{EA}{L/c} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x2} \\ \bar{u}_{y2} \end{bmatrix}$$

$$\boxed{K^1 = (T^1)^T \bar{K}^1 T^1}$$

$$\sin\varphi = \cos\alpha$$
$$\cos\varphi = -\sin\alpha$$

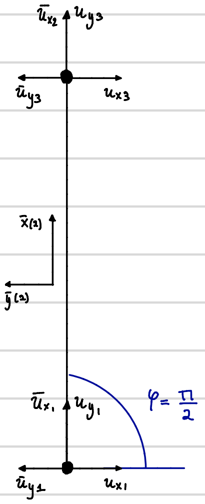
$$T^1 = \begin{bmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix}$$

$$K^1 = \frac{E^1 A^1}{L^1/c} \begin{bmatrix} s^2 & -sc & -s^2 & cs \\ -sc & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -sc & c^2 \end{bmatrix}$$

Globalized Element Stiffness Equations

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} cs^2 & -sc^2 & -s^2c & c^2s \\ -sc^2 & c^3 & c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -sc^2 \\ c^2s & -c^3 & -sc^2 & c^3 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix}$$

Element (2)



$$\begin{bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1} \\ \bar{f}_{x3} \\ \bar{f}_{y3} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x3} \\ \bar{u}_{y3} \end{bmatrix}$$

$$\boxed{K^2 = (T^2)^T \bar{K}^2\, T^2}$$

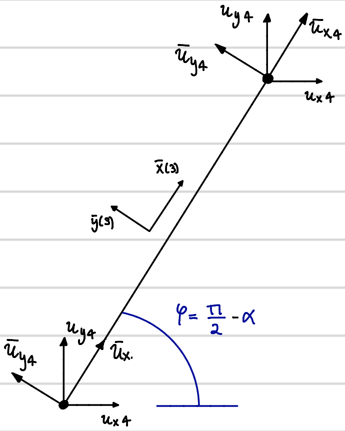
$$T^1 = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

$$K^2 = \frac{E^2 A^2}{L^2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Globalized Element STiffness Equations

$$\begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x1}^{(2)} \\ u_{y1}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix}$$

Element (3)



$$\begin{bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1} \\ \bar{f}_{x4} \\ \bar{f}_{y4} \end{bmatrix} = \frac{EA}{L/c} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x4} \\ \bar{u}_{y4} \end{bmatrix}$$

$$\boxed{K^3 = (T^3)^T \bar{K}^3\, T^3}$$

$\sin\varphi = \cos\alpha$

$\cos\varphi = \sin\alpha$

$$T^3 = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix}$$

$$K^3 = \frac{E^3 A^3}{L^3/c} \begin{bmatrix} s^2 & sc & -s^2 & -cs \\ sc & c^2 & -cs & -c^2 \\ -s^2 & -cs & s^2 & cs \\ -cs & -c^2 & sc & c^2 \end{bmatrix}$$

Globalized Element STiffness Equations

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x4}^{(3)} \\ f_{y4}^{(3)} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} cs^2 & sc^2 & -s^2c & -c^2s \\ sc^2 & c^3 & -c^2s & -c^3 \\ -cs^2 & -c^2s & cs^2 & sc^2 \\ -c^2s & -c^3 & sc^2 & c^3 \end{bmatrix} \begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x4}^{(3)} \\ u_{y4}^{(3)} \end{bmatrix}$$

After apply compatibility

$$f^{(1)} = K^{(1)} u$$

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \\ f_{x4}^{(1)} \\ f_{y4}^{(1)} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} cs^2 & -sc^2 & -s^2c & c^2s & 0 & 0 & 0 & 0 \\ -sc^2 & c^3 & c^2s & -c^3 & 0 & 0 & 0 & 0 \\ -cs^2 & c^2s & cs^2 & -sc^2 & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -sc^2 & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

$$f^{(2)} = K^{(2)} u$$

$$\begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \\ f_{x4}^{(2)} \\ f_{y4}^{(2)} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

$$f^{(3)} = K^{(3)} u$$

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x2}^{(3)} \\ f_{y2}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \\ f_{x4}^{(3)} \\ f_{y4}^{(3)} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} cs^2 & sc^2 & 0 & 0 & 0 & 0 & -s^2c & -c^2s \\ sc^2 & c^3 & 0 & 0 & 0 & 0 & -c^2s & -c^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -s^2c & -c^2s & 0 & 0 & 0 & 0 & cs^2 & sc^2 \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & sc^2 & c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Master Stiffness Equations (Equilibrium Rule)

$$f = f^{(1)} + f^{(2)} + f^{(3)} = \left(K^{(1)} + K^{(2)} + K^{(3)}\right) u = K u$$

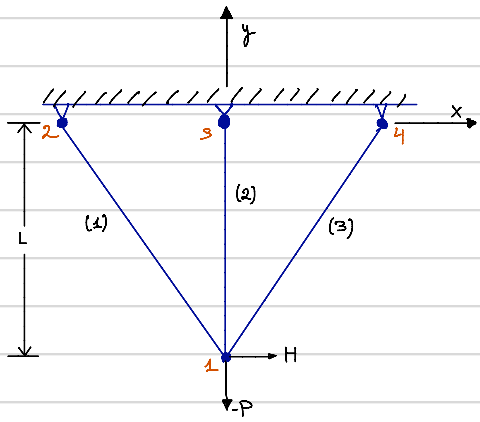
$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -s^2c & c^2s & 0 & 0 & -s^2c & -c^2s \\ 0 & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -sc^2 & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -sc^2 & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -s^2c & -c^2s & 0 & 0 & 0 & 0 & cs^2 & sc^2 \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & sc^2 & c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

The corresponding 5th row and column are the values for ux3. The reason is due to node 3 is not taking any horizontal force (mainly taken by the nodes 2 and 4).

(b) Apply the BCs and show the 2-equation modified stiffness system

Displacement BCs $\rightarrow$ $\boxed{u_{x2} = u_{y2} = u_{x3} = u_{y3} = u_{x4} = u_{y4} = 0}$

Force BCs $\rightarrow$ $\boxed{f_{x1} = H \quad , f_{y1} = -P}$



$$\begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -s^2c & c^2s & 0 & 0 & -s^2c & -c^2s \\ 0 & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -sc^2 & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -sc^2 & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -s^2c & -c^2s & 0 & 0 & 0 & 0 & cs^2 & sc^2 \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & sc^2 & c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

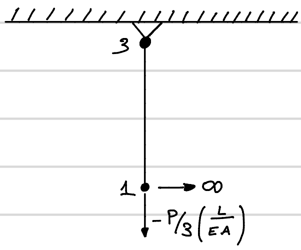
$$H = \frac{EA}{L}\left(2+cs^2\right) u_{x1}$$

$$-P = \frac{EA}{L}\left(1+2c^3\right) u_{y1}$$

(c) Solve for the displacements ux1 and uy1. Check that the solution makes physical sense for the limit cases alpha = 0 and alpha = 90. Why does ux1 "blow up" if H is different than 0 and alpha = 0?

$$u_{x1} = \frac{H}{2cs^2}\,\frac{L}{EA} \qquad \Bigg| \qquad u_{y1} = \frac{-P}{1+2c^3}\,\frac{L}{EA}$$

for $\alpha = 0$



for $\alpha = 0$

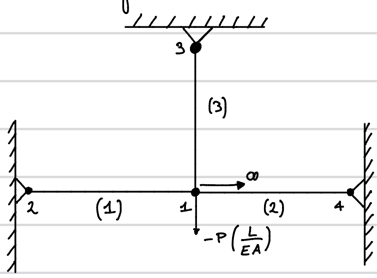
$$u_{x1} = \frac{H}{0}\,\frac{L}{EA} \qquad \Bigg| \qquad u_{y1} = \frac{-P}{3}\,\frac{L}{EA}$$

The horizontal displacement "blow up" due to the lack of equilibrium of momentum around node 3.

for $\alpha = 90°$

$$u_{x1} = \frac{H}{0}\,\frac{L}{EA} \qquad \Bigg| \qquad u_{y1} = -P\,\frac{L}{EA}$$

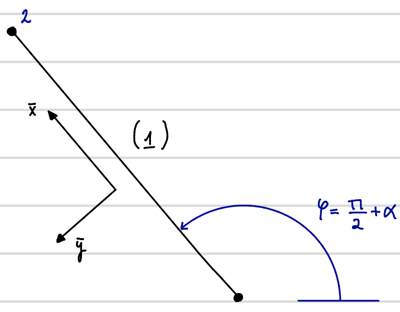
for $\alpha = 90°$



The solution has lack of physical meaning due to the infinite value for horizontal displacement. Indeed the horizontal displacement is constrained by bar elements (1) and (2).

(d) Recover the axial forces in the three members. Why F1 and F3 “blow up” if H is different than 0 and alpha = 0?

Axial forces in element (1)



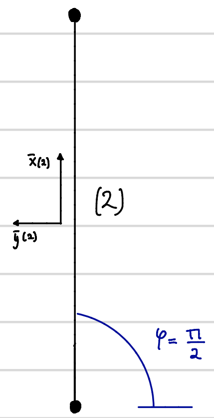
$$u^1 = \begin{bmatrix} u_{x1} & u_{y1} & u_{x2} & u_{y2} \end{bmatrix}^T = \begin{bmatrix} \frac{H}{2cs^2} & \frac{-P}{1+2c^3} & 0 & 0 \end{bmatrix}^T \frac{L}{EA}$$

$$\bar{u}^1 = T^1 u^1 \rightarrow \begin{bmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix} \begin{bmatrix} \frac{H}{2cs^2} \\ \frac{-P}{1+2c^3} \\ 0 \\ 0 \end{bmatrix} \frac{L}{EA} = \begin{bmatrix} \frac{-Hs}{2cs^2} - \frac{Pc}{1+2c^3} \\ \frac{-Hc}{2cs^2} + \frac{Ps}{1+2c^3} \\ 0 \\ 0 \end{bmatrix} \frac{L}{EA}$$

$$d^1 = \bar{u}^1_{x2} - \bar{u}^1_{x1} = 0 + \left(\frac{Hs}{2cs^2} + \frac{Pc}{1+2c^3} \right) \frac{L}{EA}$$

$$F^1 = \frac{EA}{L} \left(\frac{H\cancel{cs}}{2\cancel{cs}^2} + \frac{Pc^2}{1+2c^3} \right) = \boxed{\frac{H}{2s} + \frac{Pc^2}{1+2c^3}}$$

Axial forces in element (2)



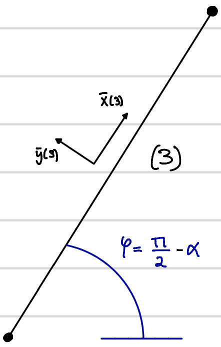
$$u^2 = \begin{bmatrix} u_{x1} & u_{y1} & u_{x3} & u_{y3} \end{bmatrix}^T = \begin{bmatrix} \frac{H}{2cs^2} & \frac{-P}{1+2c^3} & 0 & 0 \end{bmatrix}^T \frac{L}{EA}$$

$$\bar{u}^2 = T^2 u^2 \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{H}{2cs^2} \\ \frac{-P}{1+2c^3} \\ 0 \\ 0 \end{bmatrix} \frac{L}{EA} = \begin{bmatrix} \frac{H}{\cancel{2cs^2}} \quad \frac{-P}{1+2c^3} \\ \frac{-H}{2cs^2} + \frac{P}{\cancel{1+2c^3}} \\ 0 \\ 0 \end{bmatrix} \frac{L}{EA}$$

$$d^1 = \bar{u}^1_{x3} - \bar{u}^1_{x1} = 0 + \left(\frac{-Hc}{2cs^2} + \frac{Ps}{1+2c^3} \right) \frac{L}{EA}$$

$$F^2 = \frac{EA}{L} \left(+ \frac{P}{1+2c^3} \right) = \boxed{\frac{P}{1+2c^3}}$$

Axial forces in element (3)



$$u^3 = \begin{bmatrix} u_{x1} & u_{y1} & u_{x4} & u_{y4} \end{bmatrix}^T = \begin{bmatrix} \frac{H}{2cs^2} & \frac{-P}{1+2c^3} & 0 & 0 \end{bmatrix}^T \frac{L}{EA}$$

$$\bar{u}^3 = T^3 u^3 \rightarrow \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} \frac{H}{2cs^2} \\ \frac{-P}{1+2c^3} \\ 0 \\ 0 \end{bmatrix} \frac{L}{EA} = \begin{bmatrix} \frac{Hs}{2cs^2} - \frac{Pc}{1+2c^3} \\ \frac{-Hc}{2cs^2} - \frac{Ps}{1+2c^3} \\ 0 \\ 0 \end{bmatrix} \frac{L}{EA}$$

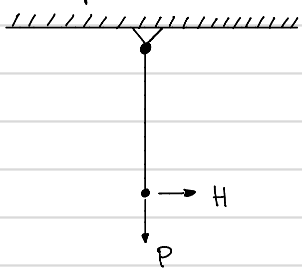
$$d^3 = \bar{u}^3_{x4} - \bar{u}^3_{x1} = 0 + \left(\frac{-Hs}{2cs^2} + \frac{Pc}{1+2c^3} \right) \frac{L}{EA}$$

$$F^3 = \frac{EA}{L} \left(\frac{-H\cancel{cs}}{2\cancel{cs}^2} + \frac{Pc^2}{1+2c^3} \right) = \boxed{\frac{-H}{2s} + \frac{Pc^2}{1+2c^3}}$$

for $\alpha = 0$ $H \neq 0$

$$F^1 = \frac{H}{0} + \frac{P}{3} \quad \Big| \quad F^3 = \frac{H}{0} + \frac{P}{3}$$

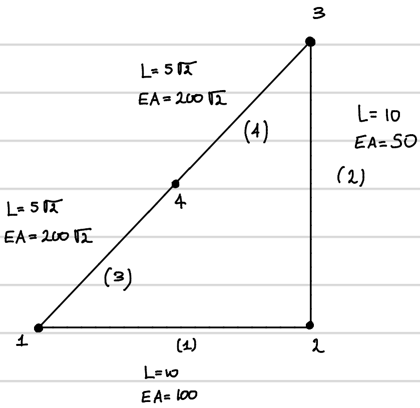
for $\alpha = 0$

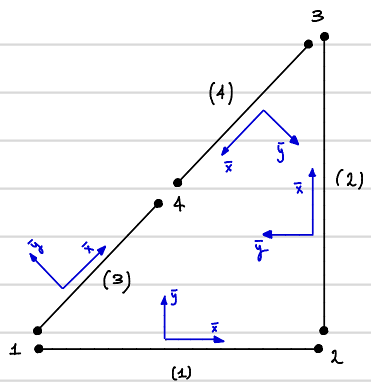


The solution for the axial forces "blow up" due to the infinite displacement in x direction.

Assignment 2

Dr. Who proposes "improving" the result for the example truss of the 1st lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr. Who suggestion by hand and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.





<u>Element 1</u>

$$\begin{bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1} \\ \bar{f}_{x2} \\ \bar{f}_{y2} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x2} \\ \bar{u}_{y2} \end{bmatrix} \longrightarrow$$

<u>Globalized Element STiffness Equations</u>

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix}$$

<u>Element 2</u>

$$\begin{bmatrix} \bar{f}_{x2} \\ \bar{f}_{y2} \\ \bar{f}_{x3} \\ \bar{f}_{y3} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{x2} \\ \bar{u}_{y2} \\ \bar{u}_{x3} \\ \bar{u}_{y3} \end{bmatrix} \longrightarrow$$

<u>Globalized Element STiffness Equations</u>

$$\begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix}$$

Element 3

Globalized Element STiffness Equations

$$\begin{bmatrix} \bar{f}_{x_1} \\ \bar{f}_{y_1} \\ \bar{f}_{x_4} \\ \bar{f}_{y_4} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{x_1} \\ \bar{u}_{y_1} \\ \bar{u}_{x_4} \\ \bar{u}_{y_4} \end{bmatrix} \longrightarrow \begin{bmatrix} f_{x_1}^{(3)} \\ f_{y_1}^{(3)} \\ f_{x_4}^{(3)} \\ f_{y_4}^{(3)} \end{bmatrix} = 40 \begin{bmatrix} 0,5 & 0,5 & -0,5 & -0,5 \\ 0,5 & 0,5 & -0,5 & -0,5 \\ -0,5 & -0,5 & 0,5 & 0,5 \\ -0,5 & -0,5 & 0,5 & 0,5 \end{bmatrix} \begin{bmatrix} u_{x_1}^{(3)} \\ u_{y_1}^{(3)} \\ u_{x_4}^{(3)} \\ u_{y_4}^{(3)} \end{bmatrix}$$

Element 4

Globalized Element STiffness Equations

$$\begin{bmatrix} \bar{f}_{x_3} \\ \bar{f}_{y_3} \\ \bar{f}_{x_4} \\ \bar{f}_{y_4} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{x_3} \\ \bar{u}_{y_3} \\ \bar{u}_{x_4} \\ \bar{u}_{y_4} \end{bmatrix} \xrightarrow[\sin\phi = -\sin 45^\circ]{\cos\phi = -\cos 45^\circ} \begin{bmatrix} f_{x_3}^{(4)} \\ f_{y_3}^{(4)} \\ f_{x_4}^{(4)} \\ f_{y_4}^{(4)} \end{bmatrix} = 40 \begin{bmatrix} 0,5 & 0,5 & -0,5 & -0,5 \\ 0,5 & 0,5 & -0,5 & -0,5 \\ -0,5 & -0,5 & 0,5 & 0,5 \\ -0,5 & -0,5 & 0,5 & 0,5 \end{bmatrix} \begin{bmatrix} u_{x_3}^{(4)} \\ u_{y_3}^{(4)} \\ u_{x_4}^{(4)} \\ u_{y_4}^{(4)} \end{bmatrix}$$

After apply compatibility

$$f^{(1)} = K^{(1)} u$$

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \\ f_{x4}^{(1)} \\ f_{y4}^{(1)} \end{bmatrix} = \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

$$f^{(2)} = K^{(2)} u$$

$$\begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \\ f_{x4}^{(2)} \\ f_{y4}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & -20 & -20 & 20 & 20 \\ 0 & 0 & 0 & 0 & -20 & -20 & 20 & 20 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

$$f^{(3)} = K^{(3)} u$$

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x2}^{(3)} \\ f_{y2}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \\ f_{x4}^{(3)} \\ f_{y4}^{(3)} \end{bmatrix} = \begin{bmatrix} 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -20 & -20 & 0 & 0 & 0 & 0 & 20 & 20 \\ -20 & -20 & 0 & 0 & 0 & 0 & 20 & 20 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

$$f^{(4)} = K^{(4)} u$$

$$\begin{bmatrix} f_{x1}^{(4)} \\ f_{y1}^{(4)} \\ f_{x2}^{(4)} \\ f_{y2}^{(4)} \\ f_{x3}^{(4)} \\ f_{y3}^{(4)} \\ f_{x4}^{(4)} \\ f_{y4}^{(4)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & -20 & -20 & 20 & 20 \\ 0 & 0 & 0 & 0 & -20 & -20 & 20 & 20 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Master Stiffness Equations (Equilibrium Rule)

$$f = f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)} = \left(K^{(1)} + K^{(2)} + K^{(3)} + K^{(4)} \right) u = K u$$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & 20 & 25 & -20 & -20 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Displacement BCs $\rightarrow$ $\boxed{u_{x_1} = u_{y_1} = u_{y_2} = 0}$

Force BCs $\rightarrow$ $\boxed{f_{x_3} = 2 \quad , \; f_{y_3} = 1}$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & 20 & 25 & -20 & -20 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_{x2} \\ 0 \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Modified stiffness matrix.

$$\begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & -20 & -20 \\ 0 & 20 & 25 & -20 & -20 \\ 0 & -20 & -20 & 40 & 40 \\ 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Singular matrix $\rightarrow$ det = 0 (No inverse)

From the physical point, we are introducing a pinned joint in the element (3). That introduces an instability to the hole structure.