Homework 1b.

1

$$x_1 = X_1 + \mathrm{sen}(\alpha) X_2$$
$$x_2 = \cos(\alpha) X_2$$

Deformation mapping $\Rightarrow \vec{x} = \vec{\varphi}(\vec{X}, t) = \begin{bmatrix} X_1 + \mathrm{sen}(\alpha) X_2 \\ \cos(\alpha) X_2 \end{bmatrix}$

2

- Deformation gradient:

$F = \nabla \vec{\varphi} \Rightarrow$

$$F = \begin{bmatrix} \frac{\partial \varphi_{x_1}}{\partial X_1} & \frac{\partial \varphi_{x_1}}{\partial X_2} \\ \frac{\partial \varphi_{x_2}}{\partial X_2} & \frac{\partial \varphi_{x_2}}{\partial X_2} \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{sen}(\alpha) \\ 0 & \cos(\alpha) \end{bmatrix}$$

- Cauchy-Green deformation tensor:

$$C = F^T F = \begin{bmatrix} 1 & \mathrm{sen}(\alpha) \\ \mathrm{sen}(\alpha) & \cos(\alpha)^2 + \mathrm{sen}(\alpha)^2 \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{sen}(\alpha) \\ \mathrm{sen}(\alpha) & 1 \end{bmatrix}$$

3

$$dv = J\,dV$$
$$\Downarrow$$
$$v = JV$$

Where:

$$J = \mathrm{def}(F) = \cos\alpha$$

$\therefore$

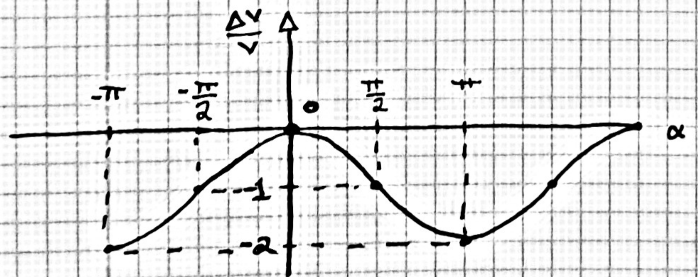
$$v = \cos(\alpha) V$$

Variation of volume:

$$\Delta V = v - V = \cos(\alpha) V - V = V(\cos(\alpha) - 1)$$

$\therefore$

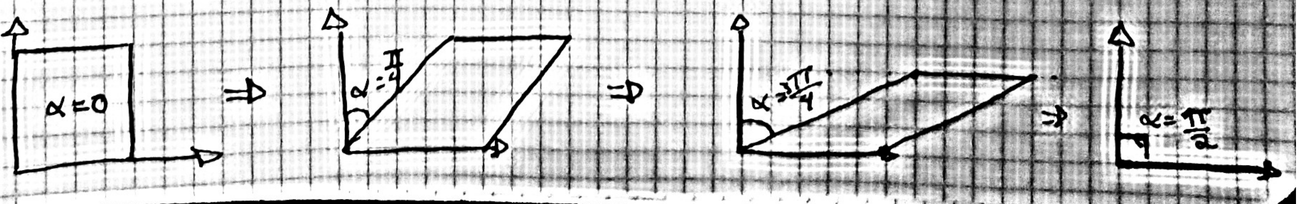
$$\frac{\Delta V}{V} = \cos(\alpha) - 1$$



4 Deformations cease to be admissible whenever the Jacobian $J \leq 0$.

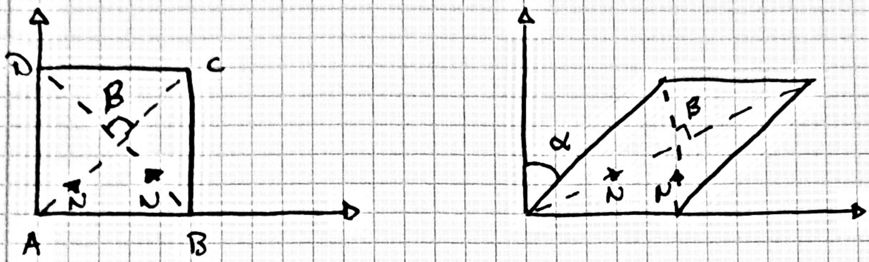
Therefore, $\Rightarrow \mathrm{abs}(\alpha) < \frac{\pi}{2}$

Geometric interpretation:



When $\alpha = \frac{\pi}{2}$ there is no possible physical way of having a volume represented in 2D.

$\boxed{5}$



$$\lambda = \sqrt{C_{IJ} N_I N_J}$$

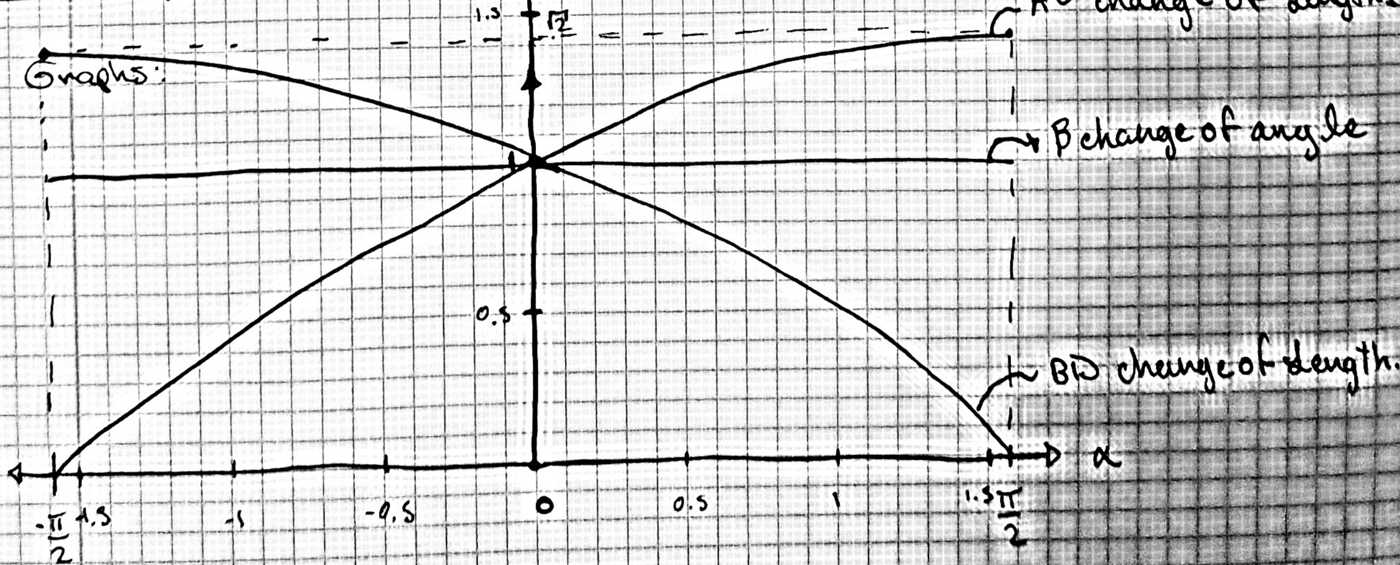
↳ Material unit vector

$$\lambda_1 = \sqrt{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & sen(\alpha) \\ sen(\alpha) & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}} \boxed{= \sqrt{\sqrt{2}\left(\frac{\sqrt{2}\,sen(\alpha)}{2} + \frac{\sqrt{2}}{2}\right)}}$$

$$\lambda_2 = \sqrt{\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & sen(\alpha) \\ sen(\alpha) & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}} \boxed{= \sqrt{-\sqrt{2}\left(\frac{\sqrt{2}\,sen(\alpha)}{2} - \frac{\sqrt{2}}{2}\right)}}$$

$$\cos\beta = \frac{C_{IJ}\, dX_I\, dY_J}{\sqrt{C_{KL}\, dX_K\, dX_L}\sqrt{C_{MN}\, dY_M\, dY_N}} = \frac{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & sen(\alpha) \\ sen(\alpha) & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}}{\sqrt{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & sen(\alpha) \\ sen(\alpha) & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}} \sqrt{\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & sen(\alpha) \\ sen(\alpha) & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}}}$$

$$\boxed{\cos\beta = 0}$$



Homework 1c

1 a) Deformation gradient field F:

$$\underline{\underline{F}} = \begin{bmatrix} \frac{\partial \varphi_1}{\partial x^1} & \frac{\partial \varphi_1}{\partial x^2} & \frac{\partial \varphi_1}{\partial x^3} \\ \frac{\partial \varphi_2}{\partial x^1} & \frac{\partial \varphi_2}{\partial x^2} & \frac{\partial \varphi_2}{\partial x^3} \\ \frac{\partial \varphi_3}{\partial x^1} & \frac{\partial \varphi^3}{\partial x^2} & \frac{\partial \varphi_3}{\partial x^3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & 1 \end{bmatrix}$$

Cauchy-Green deformation tensor C:

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \begin{bmatrix} 1 + \left(\frac{\partial w}{\partial x^1}\right)^2 & \left(\frac{\partial w}{\partial x^1}\right)\left(\frac{\partial w}{\partial x^2}\right) & \left(\frac{\partial w}{\partial x^1}\right) \\ \left(\frac{\partial w}{\partial x^1}\right)\left(\frac{\partial w}{\partial x^2}\right) & \left(\frac{\partial w}{\partial x^2}\right)^2 + 1 & \left(\frac{\partial w}{\partial x^2}\right) \\ \left(\frac{\partial w}{\partial x^1}\right) & \frac{\partial w}{\partial x^2} & 1 \end{bmatrix}$$

Jacobian J:

$$J = \det(\underline{F}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & 1 \end{vmatrix} = 1$$

b) Since $J = 1 \Rightarrow$ The deformation map φ is isochoric or volume-preserving.

c) As the Jacobian is defined positive, there exists a unique transformation and the local impenetrability condition is satisfied.

2

$$A = \frac{w_{,1} E_1 + w_{,2} E_2}{\sqrt{w_{,1}^2 + w_{,2}^2}} \qquad B = \frac{-w_{,2} E_1 + w_{,1} E_2}{\sqrt{w_{,1}^2 + w_{,2}^2}}$$

a) If $w(x^1, x^2)$ is a contour level $\Rightarrow \nabla w = w_{,1} E_1 + w_{,2} E_2$

The module is $\Rightarrow \|\nabla w\| = \sqrt{w_{,1}^2 + w_{,2}^2}$

$\therefore$

$\underline{A} = \frac{\nabla w}{\|\nabla w\|}$ = Unit vector over the contour level.

If we compute $A \cdot B = 0 \Rightarrow$ We know A and B are perpendicular.
Therefore B is a unit tangent vector.

(b)

$$\frac{da}{dA} = \sqrt{N_A C N_A^T}$$

$$N_A = A$$

$\therefore$

$$\frac{da}{dA} = \frac{1}{\sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2}} \cdot \sqrt{\left[\frac{\partial w}{\partial x^1}, \frac{\partial w}{\partial x^2}, 0\right] \begin{bmatrix} 1+\left(\frac{\partial w}{\partial x^1}\right)^2 & \left(\frac{\partial w}{\partial x^1}\right)\left(\frac{\partial w}{\partial x^2}\right) & \left(\frac{\partial w}{\partial x_1}\right) \\ \left(\frac{\partial w}{\partial x^1}\right)\left(\frac{\partial w}{\partial x^2}\right) & 1+\left(\frac{\partial w}{\partial x^2}\right) & \left(\frac{\partial w}{\partial x^2}\right) \\ \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x^1} \\ \frac{\partial w}{\partial x^2} \\ 0 \end{bmatrix}}$$

$$\frac{da}{dA} = \frac{\sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2 + 1} \cdot \sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2}}{\sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2}} = \boxed{\sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2 + 1}}$$

Now for:

$$\frac{db}{dB} = \sqrt{N_B C_B N_B^T} = \frac{\sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2}}{\sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2}} = \boxed{1}$$

Angle calculation:

$$\cos(\theta) = \frac{N_B C N_A^T}{\left(\frac{da}{dA}\right)\left(\frac{db}{dB}\right)}$$

Where: $N_B C N_A^T = 0$

Therefore:

$$\cos(\theta) = \frac{0}{\sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2}} = \boxed{0 \Rightarrow \theta = 90^\circ}$$

3

$$\underbrace{d\underline{s}_i}_{\text{spatial differential area}} = \underbrace{J}_{\text{Jacobian}}\; \underbrace{F^{-T}_{Ii}}_{\text{Inverse of the transpose of the deformation gradient tensor}} \underbrace{N_I\, d\underline{S}}_{\text{material differential area}}$$

Where:

For isochoric deformation $\Rightarrow J = 1$ and the material unit vector for $d\underline{S}$ is:

$$N_I = [0, 0, 1]$$

$$\boxed{d\underline{s} = 1 \begin{bmatrix} 1 & 0 & -\frac{\partial w}{\partial x^1} \\ 0 & 1 & -\frac{\partial w}{\partial x^2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dS = \begin{bmatrix} -\frac{\partial w}{\partial x^1} \\ -\frac{\partial w}{\partial x^2} \\ 1 \end{bmatrix} dS}$$

5

$$\boxed{\int_{\Omega} ds = \int_{\Omega_0} \sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2 + 1}\; dS}$$

6

Arc-Length of a boundary

$$\int_{d\Omega} dl = \int_{d\Omega_0} \lambda\, dL = \int_{d\Omega_0} \sqrt{1 + 2TET^T}\, dL$$

Where:

- T is the unit vector along the material direction, defined as following:

$$T = \left[\frac{x^1(s)}{ds}, \frac{x^2(s)}{ds}, 0 \right]$$

- E is the Green-Lagrange strain tensor, defined as follows:

$$E = \frac{1}{2}(C - G)$$

Where:

- G is a metric tensor, defined as follows:

$$G = I$$

Therefore, computing first E:

$$E = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial w}{\partial x^1}\right)^2 & \frac{\partial w}{\partial x^1}\frac{\partial w}{\partial x^2} & \frac{\partial w}{\partial x^1} \\ \frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x^2} & \left(\frac{\partial w}{\partial x^2}\right)^2 & \frac{\partial w}{\partial x^2} \\ \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & 0 \end{bmatrix}$$

Now, substituting into

$$\sqrt{1+2TET^T}$$

We obtain:

$$= \sqrt{1 + 2\left[\frac{x^1(s)}{\partial s}, \frac{x^2(s)}{\partial s}, 0\right]\begin{bmatrix} \left(\frac{\partial w}{\partial x^1}\right)^2 & \frac{\partial w}{\partial x^1}\frac{\partial w}{\partial x^2} & \frac{\partial w}{\partial x^1} \\ \frac{\partial w}{\partial x^1}\frac{\partial w}{\partial x^2} & \left(\frac{\partial w}{\partial x^2}\right)^2 & \frac{\partial w}{\partial x^2} \\ \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & 0 \end{bmatrix}\begin{bmatrix} \frac{x^1(s)}{ds} \\ \frac{x^2(s)}{ds} \\ 0 \end{bmatrix}}$$

$$= \sqrt{\left(\frac{x^1(s)}{\partial s}\frac{\partial w}{\partial x^1} + \frac{x^2(s)}{ds}\frac{\partial w}{\partial x^2}\right)^2 + 1}$$

Now, integrating:

$$\boxed{\int_{d\Omega_0} dl = \int_{d\Omega_0} \sqrt{\left(\frac{x^1(s)}{ds}\frac{\partial w}{\partial x^1} + \frac{x^2(s)}{\partial s}\frac{\partial w}{\partial x^2}\right)^2 + 1}\, ds}$$