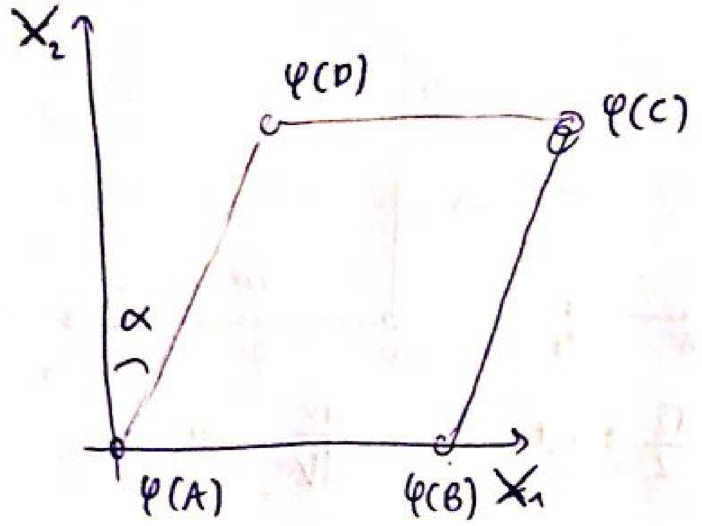
HOMEWORK B OSCAR SALVADOR



- Initially square
- sides cannot change their length

1. Deformation mapping φ

$$\varphi = \begin{Bmatrix} X_1 + \sin\alpha\, X_2 \\ \cos\alpha\, X_2 \end{Bmatrix}$$

2. $\underline{\underline{F}}$, $\underline{\underline{C}}$

$$\underline{\underline{F}} = \frac{d\underline{\varphi}}{d\underline{X}} = \boxed{\begin{bmatrix} 1 & \sin\alpha \\ 0 & \cos\alpha \end{bmatrix}}$$

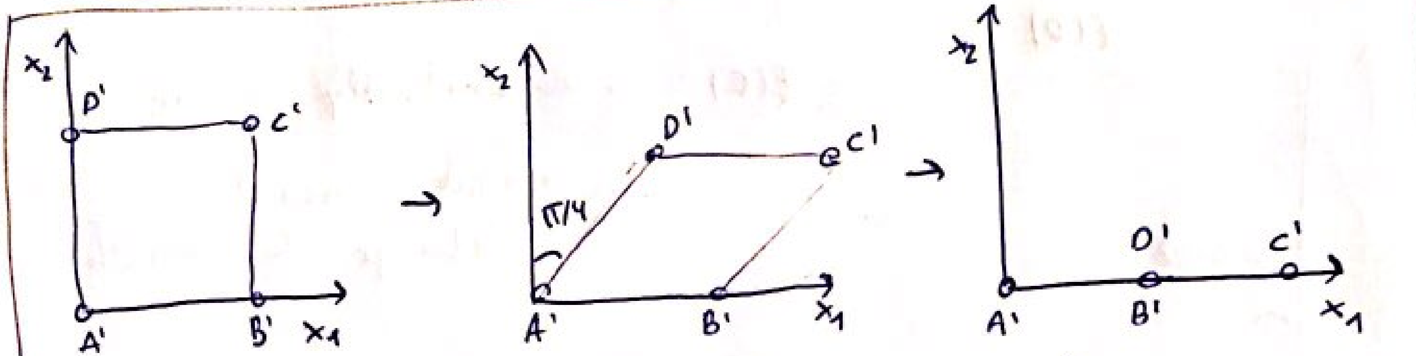
$$\underline{\underline{C}} = F^T F = \begin{bmatrix} 1 & 0 \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & \sin\alpha \\ 0 & \cos\alpha \end{bmatrix} = \boxed{\begin{bmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{bmatrix}}$$

3. Variation in volume

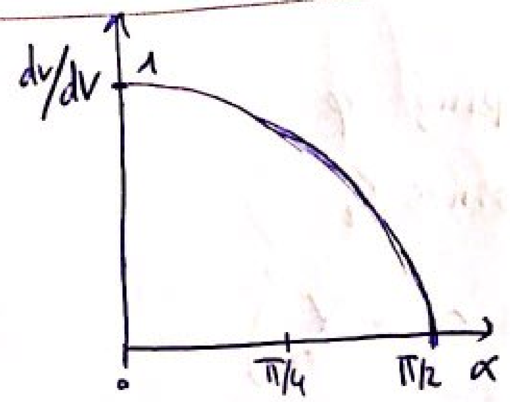
$$dv = J\, dV \qquad J = \det F$$

$$\det \underline{\underline{F}} = \cos\alpha \rightarrow \boxed{dv = \cos\alpha\, dV}$$

$\alpha = 0$ $\qquad$ $\alpha = \pi/4$ $\qquad$ $\alpha = \pi/2$ $\qquad$ *



$\frac{dv}{dV} = 1$ (No change) $\qquad$ $\frac{dv}{dV} = \frac{\sqrt{2}}{2} \simeq 0.7$ $\qquad$ $\frac{dv}{dV} = 0$



The volume continuously decreases as α increases.

4. When deformations are not admisible

If $J \leq 0$ the deformation is not admisible

In this case, as $J = \cos\alpha$, for

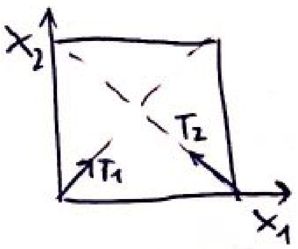
$\boxed{\alpha \geq \pi/2}$ the deformation is not admisible.

* See the geometric interpretation above

5. $d\ell$ of diagonals AC, BD

$d\beta$ being β $\widehat{AC\ BD}$

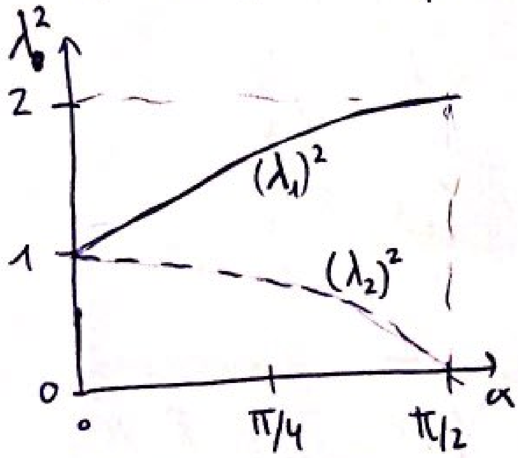
$\boxed{\text{stretching:}}$ $\lambda^2 = T \cdot C \cdot T$ being T $\begin{cases} T_1 = \frac{1}{\sqrt{2}}(1,1) \\ T_2 = \frac{1}{\sqrt{2}}(-1,1) \end{cases}$



$$\lambda_1^2 = \frac{1}{2}(1\ 1)\begin{pmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 + \sin\alpha$$

$$\lambda_2^2 = \frac{1}{2}(-1,1)\begin{pmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1 - \sin\alpha$$

Therefore, if we plot the expressions $\lambda_1(\alpha)$, $\lambda_2(\alpha)$



Here, we can see that the diagonal AC will reach a maximum when $\alpha = \pi/2$ (4 points on the horitzontal). Meanwhile, diagonal BD will ~~be minimum~~ decrease until reachs its minimum at $\alpha = \pi/2$

$\boxed{\text{Angle}}$

$$\cos\beta = \frac{T_1 \underline{\underline{C}} \cdot T_2}{\underbrace{\sqrt{T_1 \underline{\underline{C}}\, T_1}}_{\lambda_1}\ \underbrace{\sqrt{T_2 \underline{\underline{C}}\, T_2}}_{\lambda_2}}$$

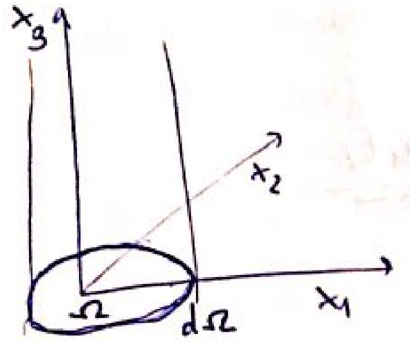
~~Bo~~ $\cos\beta = 0$ (90°)

constant

$$T_1 \cdot \underline{\underline{C}} \cdot T_2 = \frac{1}{2}(1,1)\begin{pmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 - \sin\alpha + 1 + \sin\alpha = \boxed{0}$$

OSCAR SALVADOR

HOMEWORK B C



Anti-plane shear deformation:

$$\begin{cases} \varphi_1 = X_1 \\ \varphi_2 = X_2 \\ \varphi_3 = X_3 + w(X_1, X_2) \end{cases}$$

(1) sketch deformation of Ω

a. $$\underline{\underline{F}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{pmatrix} \qquad \underline{\underline{C}} = F^T F$$

$$\underline{\underline{C}} = \begin{pmatrix} 1+\left(\frac{\partial w}{\partial X_1}\right)^2 & \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & \frac{\partial w}{\partial X_1} \\ & 1+\left(\frac{\partial w}{\partial X_2}\right)^2 & \frac{\partial w}{\partial X_2} \\ \text{symm.} & & 1 \end{pmatrix}$$

Jacobian: $J = \det F$

$$J = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{vmatrix} = 1$$

b. The volume does not change because

$dv = J\,dV$ and $J = 1$

c. The local impenetrability condition is satisfied because $\underline{\det F > 0}$.

2. Unit vectors

$$A = \frac{w_1 E_1 + w_2 E_2}{\sqrt{w_1^2 + w_2^2}} \quad , \quad B = \frac{-w_2 E_1 + w_1 E_2}{\sqrt{w_1^2 + w_2^2}}$$

a.

Assuming that $w(x_1, x_2)$ is a contour level, its gradient is: $\nabla \underline{w} = w_1 E_1 + w_2 E_2$.

So, A is a normal vector over the contour level

$$A = \frac{\nabla \underline{w}}{\| \nabla \underline{w} \|}$$

Given that $A \cdot B = 0 \rightarrow$ B is perpendicular and, therefore, is a vector contained on $w(x_1, x_2)$.

b. Change in length and angle

$$\frac{da}{d\underline{A}} = \sqrt{T_A \cdot \underline{\underline{C}} \cdot T_A} \quad \text{→ unit vector}$$

$$= \sqrt{\left(\frac{dw}{dx_1}\right)^2 + \left(\frac{dw}{dx_2}\right)^2 + 1}$$

* calculations done with the MATLAB symbolic tool

$$\frac{db}{dB} = \sqrt{T_B \cdot \underline{\underline{C}} \cdot T_B} = 1$$

For the angle:

$$\cos(\underline{A}, \underline{B}) = \frac{\sqrt{T_A \, \underline{\underline{C}} \, T_B}}{\frac{da}{dA} \frac{db}{dB}}$$

this term is equal to zero.

$\theta = 90°$

conclusions:

$\underline{A}$ and $\underline{B}$ remains orthogonal in the deformed configuration.

Regarding the length, $\underline{B}$ remains constant and $\underline{A}$ change as a function of $w(x_1, x_2)$.

The resluts coincides with the definition of ~~shear~~ antiplane shear, where the body displacements are 0 (B) but are not null in the perpendicular direction (A).

3. Piola transformation

$$d\underline{s} = J \underline{\underline{F}}^{-T} d\underline{S}$$

↳ inverse of transpose of F

the reference unit vector of $d\underline{S}$:

$$T_{dS} = [0, 0, 1]$$

$$d\underline{s} = 1 \cdot \begin{bmatrix} 1 & 0 & -dw/dx_1 \\ 0 & 1 & -dw/dx_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dS = \begin{bmatrix} -\frac{dw}{dx_1} \\ -\frac{dw}{dx_2} \\ 1 \end{bmatrix} dS$$

5. Deformed area of Ω

From the previous expression:

$$\underline{ds} = J \underline{\underline{F}}^{-T} \underline{dS} \rightarrow \int_{\Omega^0} \underline{ds} = \int_{\Omega} \| J F^{-T} dS \| d\Omega$$

$$\int_{\Omega^0} \underline{ds} = \int_{\Omega^0} \sqrt{\left(\frac{dw}{dx_1}\right)^2 + \left(\frac{dw}{dx_2}\right)^2 + 1} \, dS$$

6. boundary $\partial \Omega$:

$$x_1 = x_1(S) \qquad x_2 = x_2(S)$$

Being the boundary $\partial\Omega$

$$\begin{cases} x_1 = x_1(S) \\ x_2 = x_2(S) \end{cases} \qquad 0 \leq S \leq L \quad \text{arc-length}$$

Perimeter: φ tangent vector.

$$\cancel{\varphi = \int_0^L \sqrt{\left(\frac{dx_1(S)}{dS}\right)^2 + \left(\frac{dx_2(S)}{dS}\right)^2} dS} \qquad t = \frac{E_1 x_1(S) + E_2 x_2(S)}{\| E_1 x'(S) + E_2 x_2(S) \|}$$

$\lambda_T \rightarrow$ stretching in the direction of t

$$\varphi = \int_{\partial\Omega} \lambda_T \, dS = \int_0^L \| \underline{t} \cdot \underline{\underline{C}} \cdot \underline{t} \| dS$$