Universitat Politècnica de Catalunya

Computational Structural Mechanics

Master's Degree in Numerical Methods in Engineering

Assignments on Kinematic Strains

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Contents

Abstract

This document contains two of the proposed exercises on continuum mechanics. In it, most of the mere mechanical calculations are omitted for clarity and because the author has considered not to include them as there are available programs that can carry the calculations without any error. Instead, more weight has been given to comment the result.

1 Exercise b)

1.1 The deformation mapping in terms of α

The deformation map φ allows to compute the current or spatial configuration at time t as a function of the reference or material configuration at time $t = 0$. Since the original shape is a square and it shifts an angle α towards the positive $X1$ direction, the mapping is straightforward.

$$\mathbf{x} = \varphi(\mathbf{X}, \alpha) = \begin{bmatrix} X_1 + X_2 sin\alpha \\ X_2 cos\alpha \end{bmatrix} \tag{1}$$

It is possible to see that the deformation map fulfills the conditions, that is:

- It is continuous ans has continuous first derivative.
- Taking $t = 0$ as the reference time, where α is supposed to be zero, the reference configuration is obtained.
- It is possible to compute the inverse of the deformation map.
- It has a positive Jacobian for the range of α under study.

1.2 The deformation gradient F and the right Cauchy-Green deformation tensor C.

The Cartesian components are easily calculated through the following expressions:

$$\mathbf{F}(\mathbf{X}, \alpha) = GRAD\varphi(\mathbf{X}, \alpha) = \begin{bmatrix} \frac{\partial \varphi_1}{\partial X_1} & \frac{\partial \varphi_1}{\partial X_2} \\ \frac{\partial \varphi_2}{\partial X_1} & \frac{\partial \varphi_1}{\partial X_2} \end{bmatrix} = \begin{bmatrix} 1 & sin\alpha \\ 0 & cos\alpha \end{bmatrix} \tag{2}$$

$$\mathbf{C} = \mathbf{F^T F} = \begin{bmatrix} 1 & 0 \\ sin\alpha & cos\alpha \end{bmatrix} \begin{bmatrix} 1 & sin\alpha \\ 0 & cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & sin\alpha \\ sin\alpha & 1 \end{bmatrix} \tag{3}$$

1.3 Computation and plot of the variation in volume of the solid as a function of α

Considering that the two-dimensional solid has an initial volume of $V_0 = L^2$ where L is the side length, the differential of volumes has to satisfy$dv = JdV$. If the differential of volume is extended to all the solid, as the properties are homogeneous,

$$\frac{dv}{dV} = \frac{\Delta v}{L^2} = J = \begin{vmatrix} 1 & sin\alpha \\ 0 & cos\alpha \end{vmatrix} = cos\alpha \tag{4}$$

The following Figure presents the evolution of the Jacobian with its parametric variable.

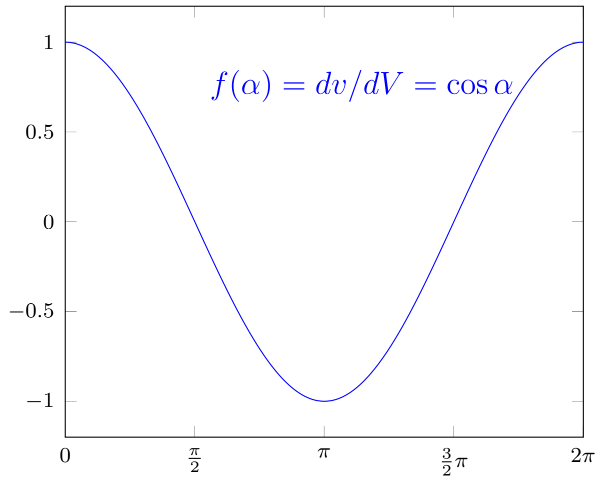

Figure 1: Plot of the volume variation againts α.

1.4 At what point do the deformations cease to be admissible? Interpret geometrically

The Jacobian has to remain positive so that the differential of volume on the deformed configuration is non-negative. A Jacobian less than zero would imply a non-physical situation. Thus, for $\alpha = \pi/2$, the solid is fully compressed in the plane $X_2 = 0$ as its differential of volume is zero. This can be seen in the plot above, where, for $\alpha = \pi/2$, $f(\alpha) = cos(\alpha) = J = 0$.

1.5 Computation of the change in length of the diagonals AC and BD, and the change in the angle β subtended by them. Interpret geometrically. Plot the change of lengths and the change of angle β as a function of α.

Let us denote the material stretch vector modulus as λ, at a material points $\mathbf{X}$ at time t, along a material direction given by the unit vector $\mathbf{T}$ on the material configuration.

$$\lambda = \sqrt{\mathbf{T} * \mathbf{CT}} \tag{5}$$

The unit vectors along the diagonals are

$$\mathbf{T_{AC}} = 1/\sqrt{2}\begin{bmatrix}1\\1\end{bmatrix} \quad \mathbf{T_{BD}} = 1/\sqrt{2}\begin{bmatrix}-1\\1\end{bmatrix} \tag{6}$$

With the previously calculated matrix $\mathbf{C}$, the obtained values are

$$\lambda_{AC} = \sqrt{1 + sin\alpha}, \quad \lambda_{BD} = \sqrt{1 - sin\alpha} \tag{7}$$

The spatial angle between the two segments at the spatial configuration is given by

$$cos\beta = \frac{\mathbf{T_{AC}} * (\mathbf{1} + 2\mathbf{E})\mathbf{T_{BD}}}{\sqrt{1 + 2\mathbf{T_{AC}} * \mathbf{E}\mathbf{T_{AC}}}\sqrt{1 + 2\mathbf{T_{BD}} * \mathbf{E}\mathbf{T_{BD}}}} \tag{8}$$

Where the unit vectors in the material configuration are the same as before and now the Green-Lagrange strain tensor must be defined as

$$\mathbf{E} = 1/2(\mathbf{C} - \mathbf{1}) = 1/2\begin{bmatrix}0 & sin\alpha\\ sin\alpha & 0\end{bmatrix} \tag{9}$$

$$\mathbf{T_{AC}} * (1 + 2\mathbf{E})\mathbf{T_{BD}} = 0 \tag{10}$$

$$\sqrt{1 + 2\mathbf{T_{AC}} * \mathbf{E}\mathbf{T_{AC}}} = \sqrt{1 + 2sin\alpha} \tag{11}$$

$$\sqrt{1 + 2\mathbf{T_{BD}} * \mathbf{E}\mathbf{T_{BD}}} = \sqrt{1 - 2sin\alpha} \tag{12}$$

Therefore $cos\beta = 0 \longrightarrow \beta = \pi/2$, which is equal to the angle between the two segments at the material configuration.

Now the following Figure exposes the evolution of the λ parameters with the angle α.

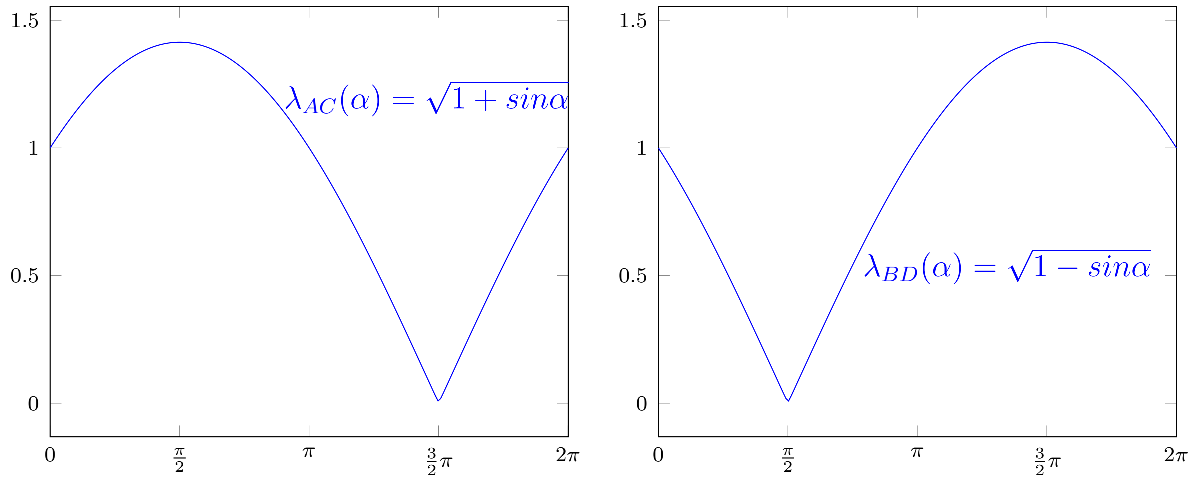

Figure 2: Plot of the modulus of the material stretch vector along AC and BD against α.

The interpretation of the results is very clear. The stretching of the diagonals are symmetric one with respect to the other. Two different cases arise:

- At the beggining, $\alpha = 0$, the two vectors do not stretch, hence the unity value. When the angle increases, the diagonal AC extends, and hence the length increases up to the physical maximum of $\alpha = \pi/2$.
- The diagonal BD is under compression once the angle starts increasing, and its length decreases until a value of zero is reached at the physical maximum when the Jacobian is zero.

2 Exercise c)

2.1 Compute the deformation gradient field F the right Cauchy-Green deformation tensor C and the Jacobian J of the deformation field in terms of w

The deformation map is the first step to compute **F** and **C**. As it is given and it fulfills the conditions:

$$\mathbf{F}(\mathbf{X},\alpha) = GRAD\varphi(\mathbf{X},\alpha) = \begin{bmatrix} \frac{\partial\varphi_1}{\partial X_1} & \frac{\partial\varphi_1}{\partial X_2} \\ \frac{\partial\varphi_2}{\partial X_1} & \frac{\partial\varphi_1}{\partial X_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{bmatrix} \tag{13}$$

The chain rule $\frac{\partial\varphi_3}{\partial X_1} = \frac{\partial\varphi_3}{\partial w}\frac{\partial w}{\partial X_1}$ has been used to compute the deformation gradient in those parts where function w is involved. Moreover, a change of notation will be used to ease the writing:

$$\begin{cases} \frac{\partial w}{\partial X_1} = u_1 \\ \frac{\partial w}{\partial X_2} = u_2 \end{cases} \tag{14}$$

Now, **C** will be:

$$\mathbf{C} = \mathbf{F^T F} = \begin{bmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_1 & u_2 & 1 \end{bmatrix} = \begin{bmatrix} 1+u_1^2 & u_2u_1 & u_1 \\ u_2u_1 & 1+u_2^2 & u_2 \\ u_1 & u_2 & 1 \end{bmatrix} \tag{15}$$

2.2 Does the solid change volume during the deformation?

As has been stated in the previous exercise, the Jacobian is associated with the volumetric deformation of the solid, and it has to remain positive so that the differential of volume on the deformed configuration is non-negative. A Jacobian less than zero would imply a non-physical situation. In this case, it is straightforward that the Jacobian is equal to 1 as $det(\mathbf{F}) = 1$. $J = 1$ implies the incompressibility condition, i.e. zero volumetric deformation, so the solid does not change its volume during the deformation.

2.3 Are the local impenetrability conditions satisfied?

Yes, they are. The Jacobian remains positive so that the deformation are in the feasible range.

2.4 How are A and B related to the level contours of w?

$$\mathbf{A} = 1/\sqrt{u_1^2 + u_2^2} \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \quad \mathbf{B} = 1/\sqrt{u_1^2 + u_2^2} \begin{bmatrix} -u_2 \\ u_1 \\ 0 \end{bmatrix} \tag{16}$$

By evaluating the vectors, it is clear that vector **A** is the normal unit vector to the contour, as it is its gradient vector, and vector **B** is the tangent vector to the curve, as it is perpendicular to **A**. It can be seen in the following picture, where the red curve is w.

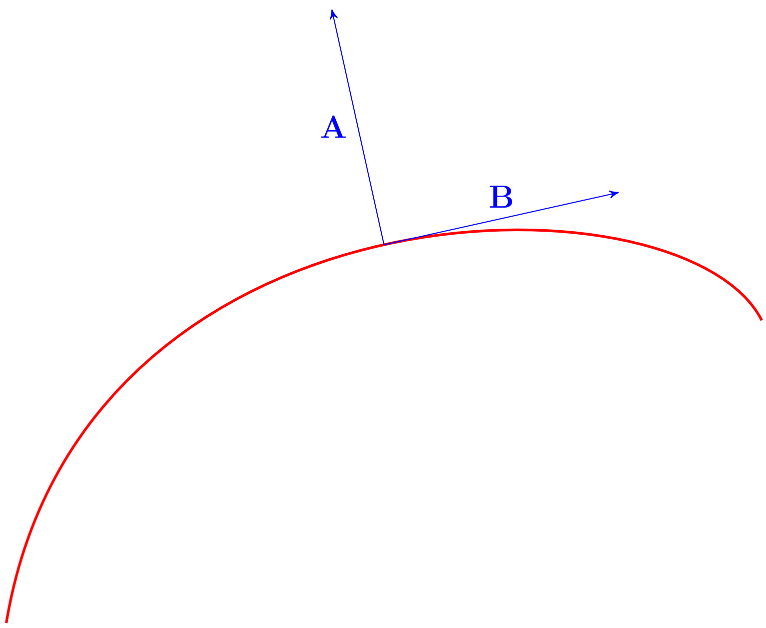

Figure 3: Relation of **A** and **B** to the level contours of w.

2.5 Compute (in terms of w) the change in length (measured by the corresponding stretch ratios) of A and B as well as the change in the angle subtended by them

The change in length of both unit vectors are computed with the known expressions in the material configuration. The unit vectors are already given and matrix **C** has been computed. The expressions obtained are given below, which have been computed with Wolfram Alpha.

$$\lambda_A = \sqrt{\mathbf{A} * \mathbf{CA}} = \sqrt{\frac{2u_1^2u_1^2 + u_1^2(1+u_1^2) + u_1^2(1+u_1^2)}{u_1^2 + u_2^2}} = \sqrt{1 + u_1^2 + u_2^2} \tag{17}$$

$$\lambda_B = \sqrt{\mathbf{B} * \mathbf{CB}} = \sqrt{\frac{-2u_1^2u_1^2 + u_1^2(1+u_2^2) + u_2^2(1+u_1^2)}{u_1^2 + u_2^2}} = 1 \tag{18}$$

The angle conformed by both vectors is

$$cos\beta = \frac{\mathbf{T_1} * (\mathbf{1} + 2\mathbf{E})\mathbf{T_2}}{\sqrt{1 + 2\mathbf{T_1} * \mathbf{ET_1}}\sqrt{1 + 2\mathbf{T_2} * \mathbf{ET_2}}} \tag{19}$$

It is only necessary to compute the numerator to realize that the angle will be zero (assuming that the denominator will be non-zero). In fact

$$\mathbf{T_1} * \mathbf{CT_2} = \frac{u_1^3 u_2 - u_1 u_2^3 - u_1 u_2(1 + u_1^2) + u_1 u_2(1 + u_2^2)}{u_1^2 + u_2^2} = 0 \tag{20}$$

Which means that both vectors will remain to be perpendicular one to the other.

2.6 Interpret the results

The results obtained regarding both unit vectors tell that both vectors remain perpendicular both in the spatial and material configurations, as the spatial angle conformed remains at $\pi/2$. On the other hand, it has been shown that the vector tangent to the contour does not stretch itself, whereas the normal vector is increasing its length at all points of Ω, as λ_A is always positive. A possible explanation is that the cylindrical solid is being compressed so that its normal cross section increases at the base.

2.7 Using the Piola transformation, compute (in terms of w) the change in area of, and in the normal to, an infinitesimal material area contained in the X_1, X_2 plane

The Piola transformation or Nanson's formula considers a differential of area vector on the reference and spatial configurations and is given by

$$d\mathbf{a} = J\mathbf{F}^{-\mathbf{T}} d\mathbf{A} \tag{21}$$

The differential of area vector, outward normal to the surface is simply $[0 \quad 0 \quad dA]^T$. Therefore,

$$d\mathbf{a} = J\mathbf{F}^{-\mathbf{T}} d\mathbf{A} = dA[-u_1 \quad -u_2 \quad 1]^T \tag{22}$$

2.8 Derive an integral expression for the deformed area of the domain Ω

The integral expression for the deformed area of the domain is the integral of the modulus of the previous result integrated over Ω.

$$\int_\Omega da = \int_\Omega \|\mathbf{da}\| = \int_\Omega \sqrt{1 + u_1^2 + u_2^2} dA \tag{23}$$

2.9 Derive an integral expression for the perimeter of the deformed boundary

Let us denote as the material stretch vector at a material point along a material direction given by the unit vector on the material configuration $\mathbf{T}$ by λ_T.

$$\int_{\partial\Omega} ds = \int_{\partial\Omega} \|\lambda_{\mathbf{T}}\| dS \tag{24}$$

Being ds the differential of arc-length defined along the perimeter. Using the vector we are given as vector $\mathbf{T}$, i.e. $\mathbf{T} = [X_1(S)/dS \quad X_2(S)/dS \quad 0]$, and together with the definition of $\lambda = \sqrt{1 + 2\mathbf{T} * \mathbf{E}\mathbf{T}}$, the integral expression is obtained. First it is necessary to define $\mathbf{E}$ as

$$\mathbf{E} = 1/2(\mathbf{C} - \mathbf{1}) = 1/2 \begin{bmatrix} u_1^2 & u_1 u_2 & u_1 \\ u_1 u_2 & u_2^2 & u_2 \\ u_1 & u_2 & 0 \end{bmatrix} \tag{25}$$

Then, operating with Wolfram Alpha the result is that

$$\lambda = \sqrt{1 + 2\mathbf{T} * \mathbf{E}\mathbf{T}} = \sqrt{1 + \left(u_1 \frac{X_1(S)}{dS} + u_2 \frac{X_s(S)}{dS} \right)^2} \tag{26}$$

Eventually,

$$\int_{\partial\Omega} ds = \int_{\partial\Omega} \sqrt{1 + \left(u_1 \frac{X_1(S)}{dS} + u_2 \frac{X_s(S)}{dS} \right)^2} dS \tag{27}$$