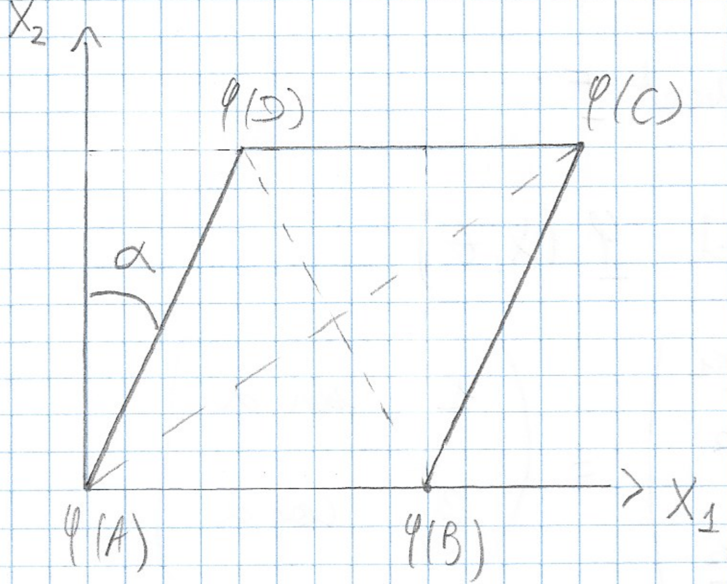
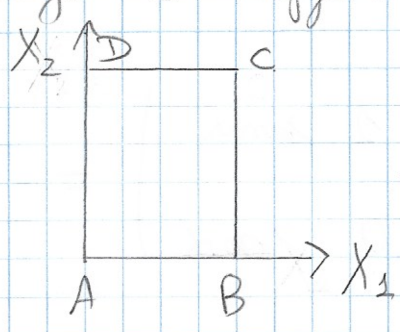
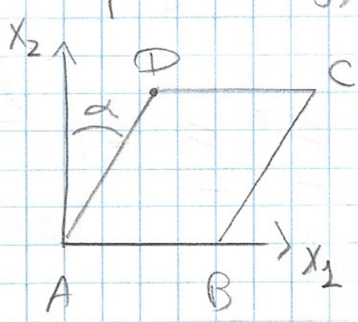
HW 1 b

Diego Roldán Uhrlep



1. Deformation mapping in terms α





$$\begin{cases} x_1 = X_1 + \sin\alpha \, X_2 \\ x_2 = \cos\alpha \, X_2 \end{cases}$$

* Deformation map:

$$\left[\underline{x} = \underline{\varphi}(X, t) = \begin{bmatrix} X_1 + \sin\alpha \, X_2 \\ \cos\alpha \, X_2 \end{bmatrix} \right]$$

2. Deformation gradient $\underline{F}$ and Cauchy-Green deformation $\underline{C}$.

* Deformation gradient $\underline{F}$:

$$\underline{F}(\underline{X}, t) = \text{Grad}\ \underline{\varphi}(\underline{X}, t)$$

$$\underline{F} = \begin{pmatrix} \dfrac{\partial \varphi_{x_1}}{\partial X_1} & \dfrac{\partial \varphi_{x_1}}{\partial X_2} \\ \dfrac{\partial \varphi_{x_2}}{\partial X_1} & \dfrac{\partial \varphi_{x_2}}{\partial X_2} \end{pmatrix} = \begin{pmatrix} 1 & \sin\alpha \\ 0 & \cos\alpha \end{pmatrix}$$

* Right Cauchy-Green deformation tensor $\underline{C}$:

$$\underline{C} = \underline{F}^T\ \underline{F} = \begin{pmatrix} 1 & 0 \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} 1 & \sin\alpha \\ 0 & \cos\alpha \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & \sin\alpha \\ \sin\alpha & \sin^2\alpha + \cos^2\alpha \end{pmatrix} = \begin{pmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{pmatrix}$$

3. Compute and plot the variation of volume as a function of α.

+ Differential Volume Map:

$$dv = J\, dV$$

+ Volumetric deformation:

$$e = \frac{dv - dV}{dV} = \frac{dv}{dV} - 1 = J - 1$$

where J is the Jacobian defined as,

$$J = \det \underline{F} = \cos \alpha$$

So, the variation of volume is:

$$\left[e = \cos \alpha - 1 \right]$$

4. At what point do the deformations cease to be admissible?

The deformations cease to be admissible when:

$$\left[J \leq 0 \right]$$

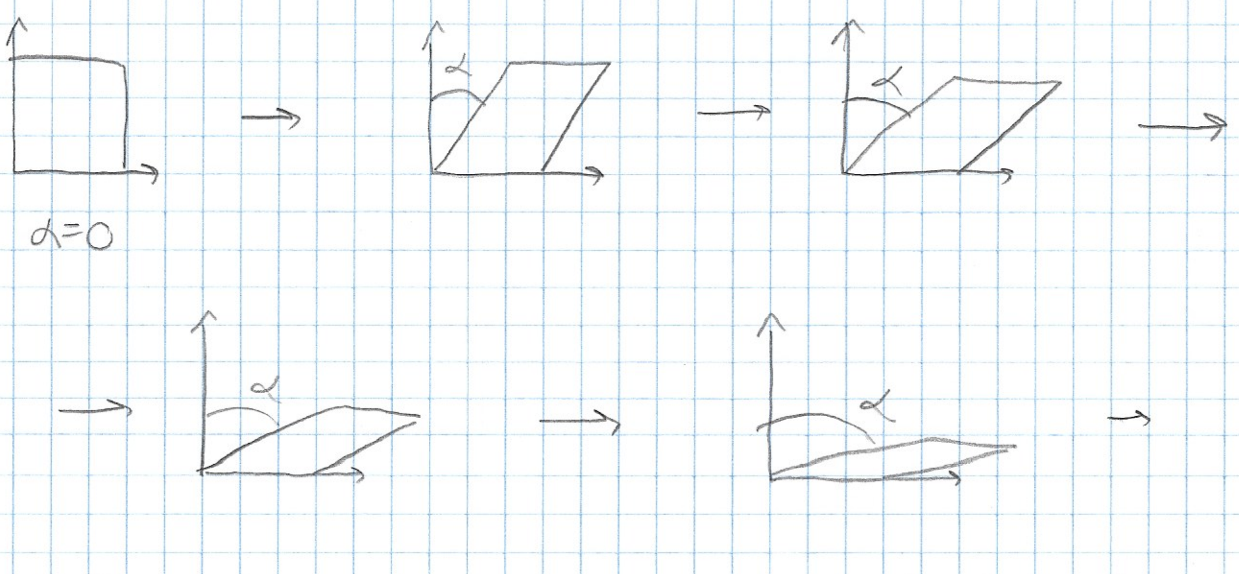
So, when

$$J = \det \underline{F} = \cos\alpha \leq 0$$

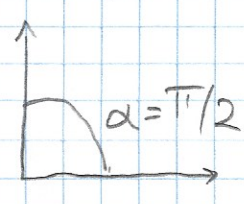
$$\left[\alpha \geq \pm \pi/2 \quad \text{the deformation is not admissible} \right]$$

$$\left[\text{Only admissible in the range: } -\pi/2 < \alpha < \pi/2 \right]$$

Geometrically:

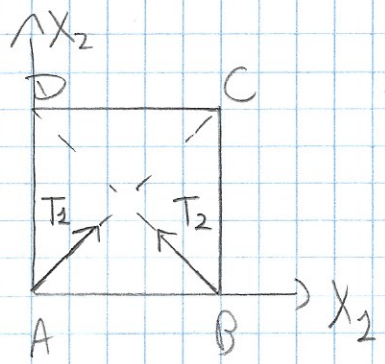


Finally when $\alpha = \pi/2$, all corners (A, B, C, D) would be in the same line, deformating the solid in an impossible physical way.



5. Change in length of diagonals AC and BD and change in angle β subtended by them. Plot the results.

+ Reference configuration:



+ Material stretch vector:

$$\lambda = \sqrt{(T \cdot CT)} \quad \text{, being } T \text{ the unit vectors}$$

$$T_1 = \frac{1}{\sqrt{2}}(1,1) \quad ; \quad T_2 = \frac{1}{2}(-1,1)$$

↳ For AC diagonal ↳ For BD diagonal

+ For diagonal AC:

$$\lambda_1 = \sqrt{\frac{1}{\sqrt{2}}(1,1) \cdot \begin{pmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \sqrt{1+\sin\alpha}$$

+ For diagonal BD:

$$\lambda_2 = \frac{1}{\sqrt{2}}(-1,1) \cdot \begin{pmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \sqrt{1-\sin\alpha}$$

From,

$$ds = \lambda\, dS \quad ; \quad \left[\lambda = \frac{ds}{dS}\right] \Rightarrow \left[s = \lambda S\right]$$

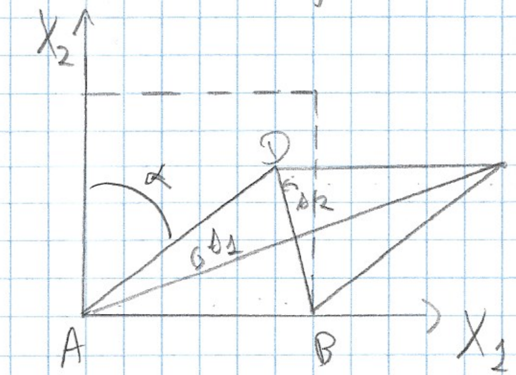
So,

$$s_1 = \lambda_1 S_1 = \sqrt{(1 + \sin\alpha)}\, S_1$$

$$s_2 = \lambda_2 S_2 = \sqrt{(1 - \sin\alpha)}\, S_2$$

being S_1 and S_2, the initial of the diagonals.

* Geometrically



When α increases,

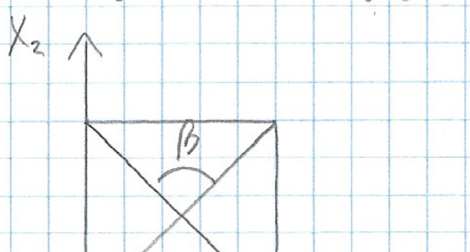
s_1 = Diagonal AC increases, because $\sqrt{1 + \sin\alpha}\, S_1$

s_2 = diag DB decreases, because $\sqrt{1 - \sin\alpha}\, S_2$

As it is possible to appreciate geometricalley.

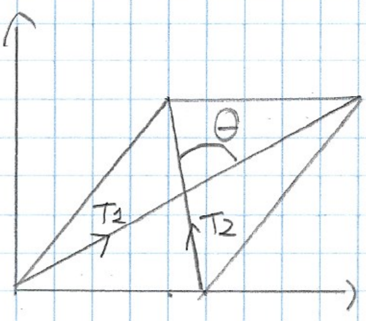
* Change of angle β:

* Reference configuration:



$\beta = \pi/2$ rad

* Spatial configuration



* Variation of angle:

$$\cos\theta = \frac{T^{(1)} \cdot (1+2E)\, T^{(2)}}{\sqrt{1+2\,T^{(1)}\cdot E\,T^{(1)}} \cdot \sqrt{1+2\,T^{(2)}\,E\,T^{(2)}}}$$

Being E the Green-Lagrange Strain Tensor:

$$E = \frac{1}{2}(C-1) = \frac{1}{2}\begin{pmatrix} 0 & \sin\alpha \\ \sin\alpha & 0 \end{pmatrix}$$

* Numerator part

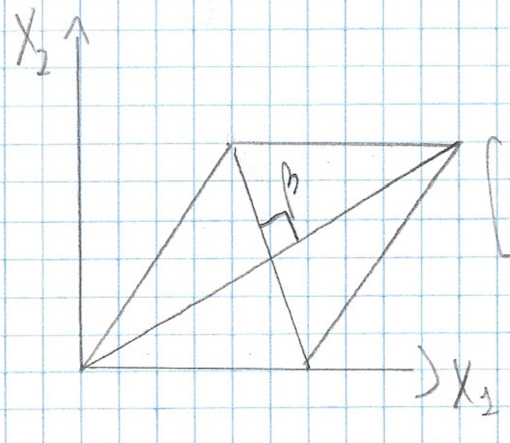
$$\frac{1}{\sqrt{2}}(1,1)\cdot\left(1+\frac{2}{2}\begin{pmatrix} 0 & \sin\alpha \\ \sin\alpha & 0 \end{pmatrix}\right)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} =$$

$$\frac{1}{2}(\sin\alpha - 1 + 1 - \sin\alpha) = 0$$

So,

$$\left[\cos\theta = 0 \quad \forall\alpha\right] ; \quad \left[\theta = \frac{\pi}{2}\ rad\right]$$

Geometrically in spatial configuration.

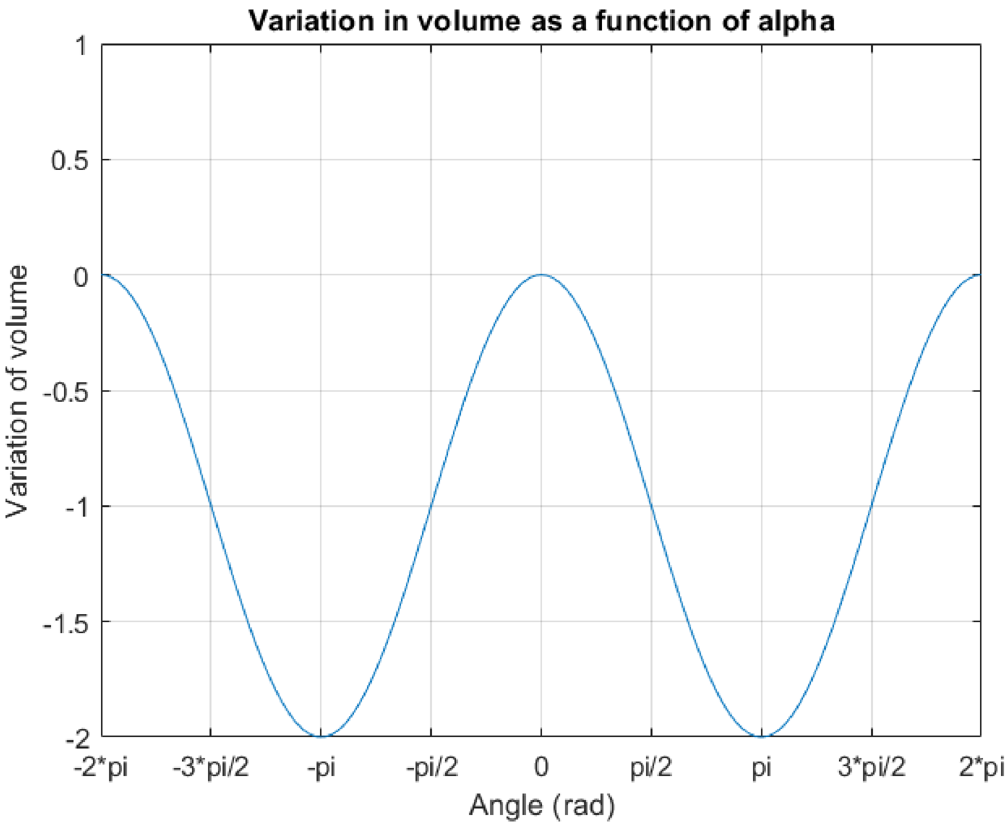


$[\beta = \pi/2 \text{ rad}]$

3. Compute and plot the variation in volume of the solid as a function of α.

Next, it presented the plot of the variation in volume of the solid as a function of α.

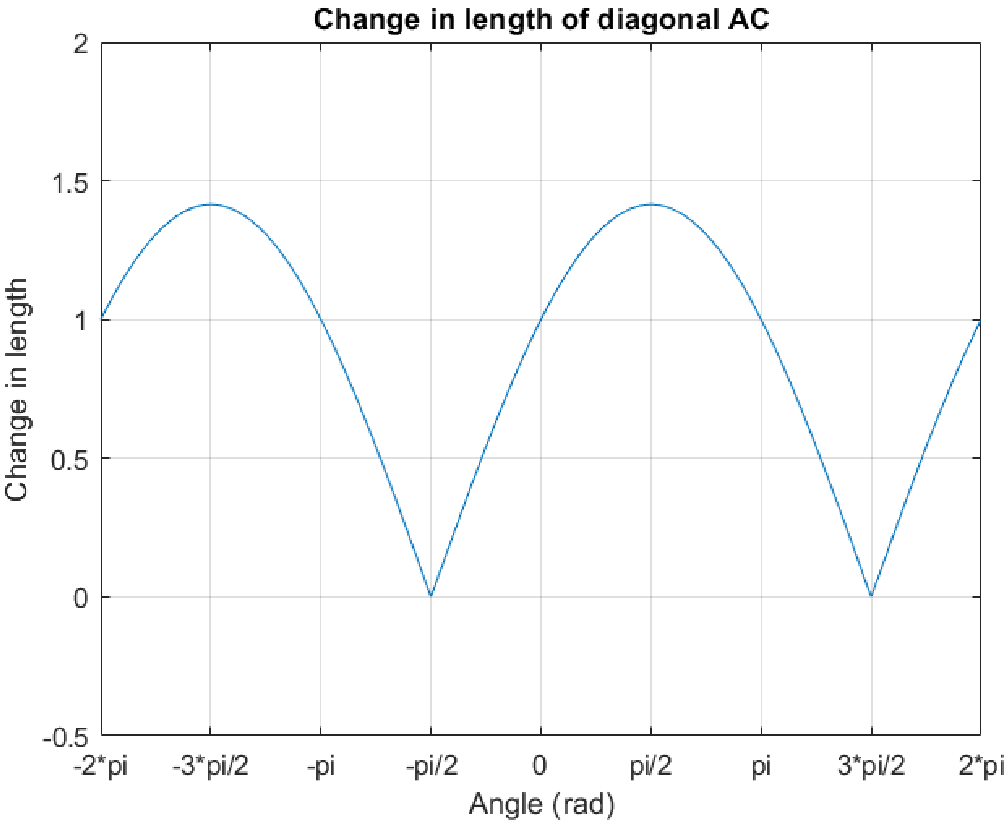
Remark that the only feasible and physical solution is between -π/2 and π/2 rad as it is mentioned before. It is plotted the rest to show the behaviour of the curve.

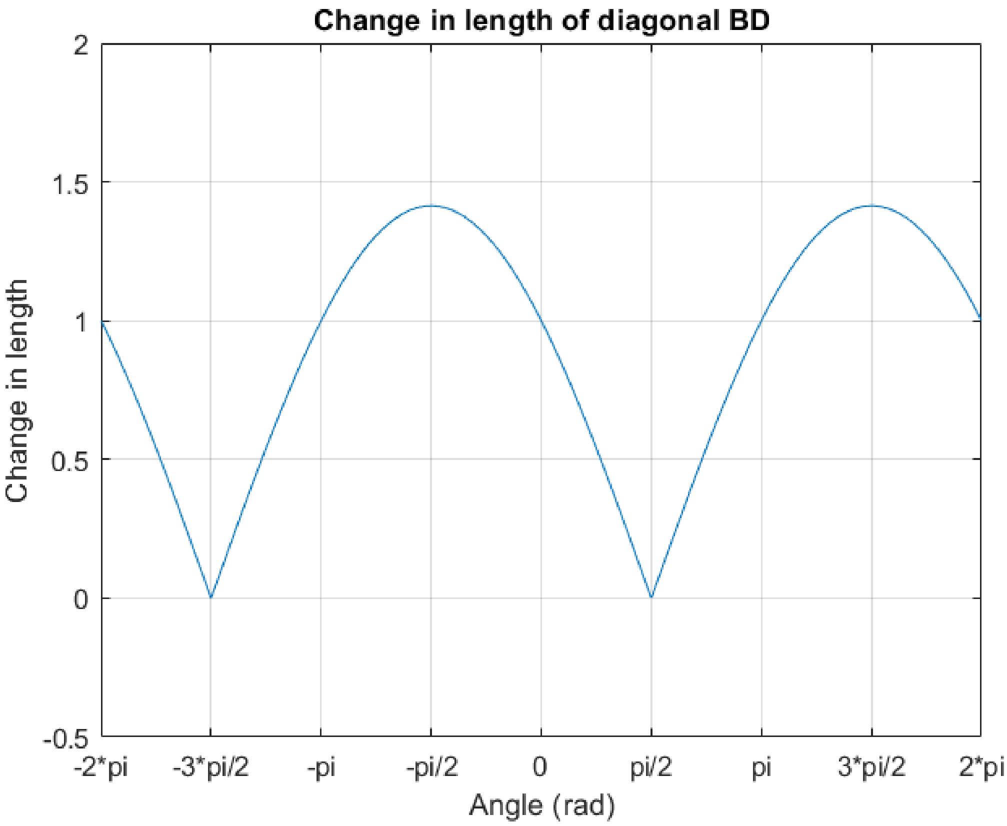


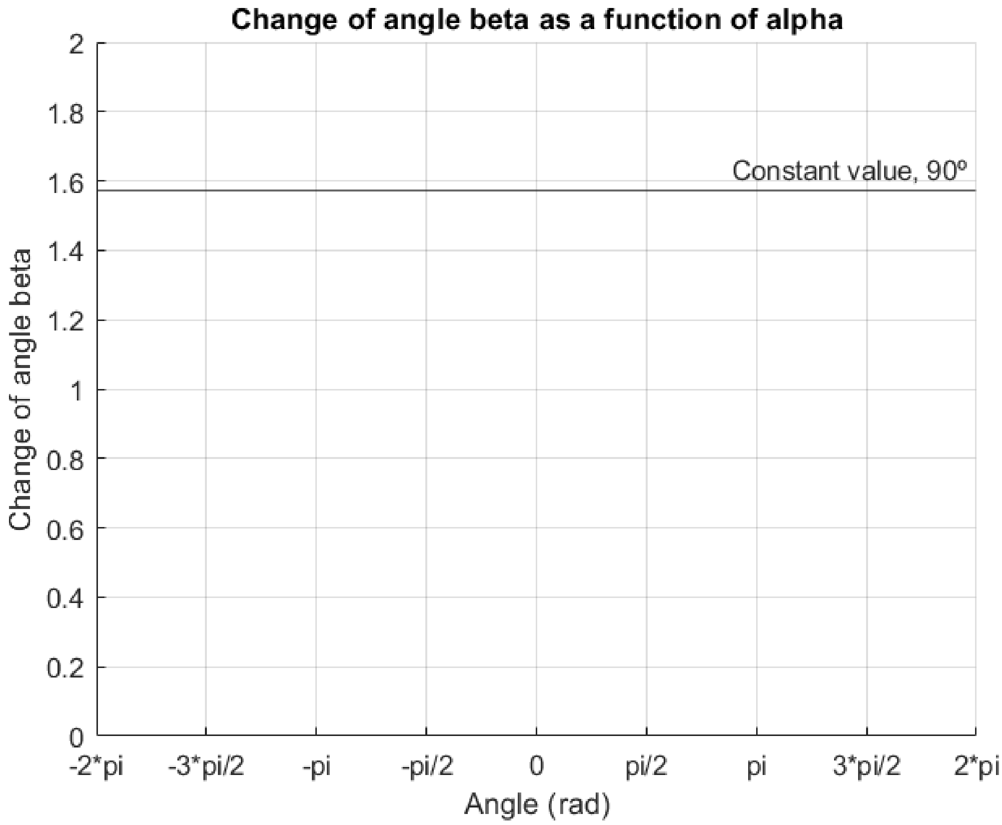
5. Compute the change in length of the diagonals AC and BD, and the change in the angle β subtended by them. Interpret geometrically. Plot the change of lengths and the change of angle β as a function of α.

Next, it presented the plot of the change of lengths and the change of angle β as a function of α.

Remark that the only feasible and physical solution is between -π/2 and π/2 rad as it is mentioned before. It is plotted the rest to show the behaviour of the curve.


Change in length of diagonal AC
Change in length
2
1.5
1
0.5
0
-0.5
-2*pi
-3*pi/2
-pi
-pi/2
0
pi/2
pi
3*pi/2
2*pi
Angle (rad)


Change in length of diagonal BD
Change in length
2
1.5
1
0.5
0
-0.5
-2*pi
-3*pi/2
-pi
-pi/2
0
pi/2
pi
3*pi/2
2*pi
Angle (rad)


Change of angle beta as a function of alpha
Constant value, 90º
Change of angle beta
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-2*pi
-3*pi/2
-pi
-pi/2
0
pi/2
pi
3*pi/2
2*pi
Angle (rad)

HW 1c

Diego Roldán Uhrlep

Deformation mapping:

$\varphi_1 = X^1 \quad ; \quad \varphi_2 = X^2 \quad ; \quad \varphi_3 = X^3 + w(X^1, X^2)$

Cylindrical solid referred to an orthonormal Cartesian reference frame $\{X^1, X^2, X^3\}$

1. a) Compute $\underline{F}$, $\underline{C}$ and J in terms of w.

Deformation gradient field $\underline{F}$:

$$\underline{F} = \begin{pmatrix} \frac{\partial \varphi_1}{\partial X^1} & \frac{\partial \varphi_1}{\partial X^2} & \frac{\partial \varphi_1}{\partial X^3} \\ \frac{\partial \varphi_2}{\partial X_1} & \frac{\partial \varphi_2}{\partial X_2} & \frac{\partial \varphi_2}{\partial X_3} \\ \frac{\partial \varphi_3}{\partial X_1} & \frac{\partial \varphi_3}{\partial X_2} & \frac{\partial \varphi_3}{\partial X_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{pmatrix}$$

Right Cauchy-Green deformation tensor $\underline{C}$:

$\underline{C} = \underline{F}^T \underline{F}$

$$\underline{C} = \begin{pmatrix} 1 & 0 & \frac{\partial w}{\partial X_1} \\ 0 & 1 & \frac{\partial w}{\partial X_2} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{pmatrix} = \begin{pmatrix} 1+\left(\frac{\partial w}{\partial X_1}\right)^2 & \frac{\partial w}{\partial X_1} \cdot \frac{\partial w}{\partial X_2} & \frac{\partial w}{\partial X_1} \\ \frac{\partial w}{\partial X_2} \cdot \frac{\partial w}{\partial X_1} & 1+\left(\frac{\partial w}{\partial X_2}\right)^2 & \frac{\partial w}{\partial X_2} \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{pmatrix}$$

Note that C is symmetric, as it was expected from definition.

Jacobean (J)

$$J = \det(\underline{F}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{vmatrix} = 1 //$$

b) Does the solid change volume during deformation?

By definition, the incompressibility condition (zero volumetric deformation), takes the form:

$$[J = 1]$$

Therefore, as it is shown previously, this solid does not change volume during deformation because it fulfils the condition. So,

$$[dv = dV] \Rightarrow [v = V]$$

c) are the local impenetrability conditions satisfied?

Due to the fact $J > 0$, it exists an inverse mapping deformation. This means that there is a spatial point related to a material point with an unique transformation.

2. Consider the unit vectors

$$\underline{A} = \frac{w_1 E_1 + w_2 E_2}{\sqrt{w_1^2 + w_2^2}} \qquad \underline{B} = \frac{-w_2 E_1 + w_1 E_2}{\sqrt{w_1^2 + w_2^2}}$$

a) How are $\underline{A}$ and $\underline{B}$ related to the level contours of $w(X_1, X_2)$.

Assuming that $w(X_1, X_2)$ is a contour level, then is constant by definition: $w(X_1, X_2) = \text{constant}$. Therefore, its gradient can be defined as:

$$\underline{\nabla w} = w_1 \underline{E_1} + w_2 \underline{E_2}$$

with the module:

$$\|\underline{\nabla w}\| = \sqrt{(w_1 E_1)^2 + (w_2 E_2)^2}$$

Then vector A is the unit normal vector over the contour level. $w(X_1, X_2) = cte$.

$$\boxed{\underline{A} = \frac{\nabla w}{\|\underline{\nabla w}\|}}$$

Finally,

$$\underline{A} \cdot \underline{B} = 0$$

So, $\underline{A}$ and $\underline{B}$ are orthogonal.

$\underline{B}$ is a unit tangent vector of $w(X_1, X_2) = cte$.

b) Compute the change in length of $\underline{A}$ and $\underline{B}$ and the angle subtended by $\underline{A}$ and $\underline{B}$.

Stretch ratio, change in length:

$$\frac{ds}{d\underline{S}} = \lambda = \sqrt{\underline{T} \cdot \underline{\underline{C}}\,\underline{T}} \quad , \text{ where } T \text{ are the unit vectors.}$$

+ For vector A: $\left(\underline{T} = \underline{A} = \frac{1}{\sqrt{\left(\frac{dw}{dX_1}\right)^2 + \left(\frac{dw}{dX_2}\right)^2}} \left(\frac{dw}{dX_1}, \frac{dw}{dX_2}, 0\right)\right)$

$$\left[\frac{ds_A}{dS_A} = \lambda_A = \sqrt{\underline{A} \cdot \underline{\underline{C}}\,\underline{A}} = \sqrt{\left(\frac{dw}{dX_1}\right)^2 + \left(\frac{dw}{dX_2}\right)^2 + 1}\right]$$

+ For vector B: $T = B = \frac{1}{\sqrt{\left(\frac{dw}{dX_1}\right)^2 + \left(\frac{dw}{dX_2}\right)^2}} \cdot \left(\frac{-dw}{dX_2}, \frac{-dw}{dX_1}, 0\right)$

$$\left[\frac{ds_B}{dS_B} = \lambda_B = \sqrt{\underline{B} \cdot \underline{\underline{C}}\,\underline{B}} = 1\right]$$

+ To work out the angle: θ = angle subtended by $\underline{A}$ and $\underline{B}$

$$\cos\theta = \frac{A \cdot (1 + 2E)B}{\sqrt{1 + 2A \cdot EA} \cdot \sqrt{1 + 2BEB}}$$

Where E is the Green-Lagrange Strain Tensor:

$$\underline{\underline{E}} = \frac{1}{2}\left(C - 1\right) = \frac{1}{2}\begin{pmatrix} \left(\frac{\partial w}{\partial X_1}\right)^2 & \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & \frac{\partial w}{\partial X_1} \\ & \left(\frac{\partial w}{\partial X_2}\right)^2 & \frac{\partial w}{\partial X_2} \\ \text{sym} & & 0 \end{pmatrix}$$

* The numerator:

$$A \cdot (1 + 2E) B = 0$$

Then,

$$\cos\theta = 0 \Rightarrow \boxed{\theta = \frac{\pi}{2} \text{ rad}}$$

* all calculations done with calculator, it is only shown the results.

c) Interpret the results

- As $\theta = 90°$, angle substended by A and B, these vectors are orthogonal in material and spatial references.

- As $\|\lambda_A\| > 1$, the vector $\underline{A}$ increases in length in spatial configuration.

  as $\|\lambda_B\| = 1$, the vector B does not change in length in spatial configuration.

- So, the normal vector (A) changes in length, whereas the tangent vector (B) remains constant.

This is related to antiplane shear which consists of having 0 displacement in the body in the plane studied and having non-zero displacement in the perpendicular direction to the plane.

3. Using Piola transformation, compute change of area.

+ The change of area is defined as:

$$d\underline{a} \cdot \underline{n} = J \cdot \underline{\underline{F}}^{-T} \underline{N} \, d\underline{A}$$

where J is the Jacobian and F^{-T} the inverse of the transpose of the deformation gradient.

$J = 1$, as it was computed before (incompressible body)

$\underline{N}$ is the normal vector $N: [0, 0, 1]$

$$\left[\frac{da}{dA} \right] = 1 \begin{bmatrix} 1 & 0 & -\frac{dw}{dX_1} \\ 0 & 1 & -\frac{dw}{dX_2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \left[\begin{bmatrix} -\frac{dw}{dX_1} \\ -\frac{dw}{dX_2} \\ 1 \end{bmatrix} \right]$$

5. Integral expression for the deformed area of the domain Ω

From previous expression:

$$da\,\underline{n} = J \cdot \underline{\underline{F}}^{-T}\,\underline{N}\,dA$$

$$\int_{\Omega} da\,\underline{n} = \int_{\Omega} \| J\,F^{-T}\,\underline{N}\,dA \|\,d\Omega =$$

$$= \int_{\Omega} da\,\underline{n} = \int_{\Omega} \left\| 1 \cdot \underline{\underline{F}}^{-T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\| dA = \int_{\Omega} \sqrt{\left(\frac{dw}{dX_1}\right)^2 + \left(\frac{dw}{dX_2}\right)^2 + 1}\; dA$$

6. Boundary $\partial\Omega$ of Ω

$$X_1 = X_1(S) \quad , \quad X_2 = X_2(S)$$

$$0 \leq S \leq L \quad \Big/ \quad \frac{\underline{E}_1 X_1(S)}{dS} + \frac{\underline{E}_2 X_2(S)}{dS} = \text{unit vector tangent to } \partial\Omega.$$

+ The arc-length is defined as:

$$\int_{\partial\Omega} dl = \int_{\partial\Omega_0} \lambda\,dL = \int_{\partial\Omega_0} \sqrt{1 + 2\,\underline{T}\,\underline{\underline{E}}\,\underline{T}}\;dL$$

where T is the unit vector along material direction.

$$\underline{T} = \begin{bmatrix} \frac{X_1(S)}{dS} & \frac{X_2(S)}{dS} & 0 \end{bmatrix}$$

And the matrix $\underline{\underline{E}}$ is defined previously.

$$\sqrt{1 + 2\underline{T}\,\underline{\underline{E}}\,\underline{T}} = \sqrt{\left(\frac{X_1(S)}{dS}\frac{dw}{dX_1} + \frac{X_2(S)}{dS}\frac{dw}{dX_2}\right)^2 + 1}$$

* Done by calculator

Then, the perimeter of the deformed boundary is:

$$\int_{d\Omega_0} dl = \int_{d\Omega_0} \sqrt{1 + 2TET} = \int_{d\Omega_0} \sqrt{\left(\frac{X_1(S)}{dS}\frac{dw}{dX_1} + \frac{X_2(S)}{dS}\frac{dw}{dX_2}\right)^2 + 1}\, dS$$