COMPUTATIONAL SOLID MEACHANICS
Master of Science in Computational Mechanics/Numerical Methods
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Assignment 3: 1st part

Kirchhoff Saint-Venant material model

Isotropic linear elasticity can be derived from balance of linear momentum, the linearized strain displacement relation $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ and the stored elastic energy function

$$W(\boldsymbol{\varepsilon}) = \frac{\lambda}{2}(\mathrm{tr}\,\boldsymbol{\varepsilon})^2 + \mu\,\mathrm{tr}(\boldsymbol{\varepsilon}^2) = \frac{\lambda}{2}(\varepsilon_{ii}^2) + \mu\,\varepsilon_{jk}\varepsilon_{jk}$$

1. Check that, the stress tensor obtained form $\boldsymbol{\sigma} = \partial W/\partial\varepsilon$ agrees with the usual linear elasticity expression.

The stress is calculated as follows:

$$\sigma_{ij} = \frac{\partial W}{\partial\varepsilon_{ij}} = \frac{\partial}{\partial\varepsilon_{ij}}\left(\frac{\lambda}{2}(\varepsilon_{ii}^2) + \mu\,\varepsilon_{jk}\varepsilon_{jk}\right) = \frac{\lambda}{2}\cdot 2\varepsilon_{ii}\delta_{ij} + 2\mu\varepsilon_{jk}\delta_{ij}\delta_{jk} = \lambda\varepsilon_{ii}\delta_{ij} + 2\mu\varepsilon_{ij}$$

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial\boldsymbol{\varepsilon}} = \lambda\,\mathrm{tr}\,\varepsilon\,\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$$

Since the linearization of the Green-Lagrange strain tensor $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$ is the small strain tensor $\boldsymbol{\varepsilon}$, it is natural to extend isotropic elasticity to nonlinear elasticity as

$$W(\boldsymbol{E}) = \frac{\lambda}{2}(\mathrm{tr}\,\boldsymbol{E})^2 + \mu\,\mathrm{tr}(\boldsymbol{E}^2).$$

This hyperelastic model is called Kirchhoff Saint-Venant material model.

2. According to the definition we gave in class about isotropy in nonlinear elasticity, is this model isotropic?

First, the energy function is expressed in terms of the deformation gradient $\boldsymbol{F}$:

$$W(\boldsymbol{F}) = \frac{\lambda}{2}\left(\frac{1}{2}\,\mathrm{tr}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})\right)^2 + \mu\,\mathrm{tr}\left(\frac{1}{4}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})^2\right)$$

Given an arbitrary rotation matrix $\boldsymbol{Q} \in SO(3)$.

$$W(\boldsymbol{FQ}) = \frac{\lambda}{2}\left(\frac{1}{2}\,\mathrm{tr}((\boldsymbol{FQ})^T\boldsymbol{FQ} - \boldsymbol{I})\right)^2 + \mu\,\mathrm{tr}\left(\frac{1}{4}((\boldsymbol{FQ})^T\boldsymbol{FQ} - \boldsymbol{I})^2\right) =$$

$$\frac{\lambda}{2}\left(\frac{1}{2}\,\mathrm{tr}((\boldsymbol{FQ})^T\boldsymbol{FQ} - \boldsymbol{I})\right)^2 + \mu\,\mathrm{tr}\left(\frac{1}{4}((\boldsymbol{FQ})^T\boldsymbol{FQ}(\boldsymbol{FQ})^T\boldsymbol{FQ} - 2(\boldsymbol{FQ})^T\boldsymbol{FQ} + \boldsymbol{I})^2\right)$$

We will use the property of the trace of a matrix multiplication:

$$\mathrm{tr}(\boldsymbol{A}^T\boldsymbol{B}) = \mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^T)$$

Applying it to the previous expression along with the fact that the trace is a linear operator:

$$W(\boldsymbol{FQ}) = \frac{\lambda}{8}\left(\operatorname{tr}(\boldsymbol{FQ}(\boldsymbol{FQ})^T) - \operatorname{tr}(\boldsymbol{I})\right)^2 + \frac{\mu}{4}\left(\operatorname{tr}((\boldsymbol{FQ})^T\boldsymbol{FQQ}^T\boldsymbol{F}^T\boldsymbol{FQ}) - 2\operatorname{tr}(\boldsymbol{FQ}(\boldsymbol{FQ})^T) + \operatorname{tr}(\boldsymbol{I})\right) =$$

$$\frac{\lambda}{8}\left(\operatorname{tr}(\boldsymbol{FQQ}^T\boldsymbol{F}^T) - \operatorname{tr}(\boldsymbol{I})\right)^2 + \frac{\mu}{4}\left(\operatorname{tr}(\boldsymbol{Q}^T\boldsymbol{F}^T\boldsymbol{FF}^T\boldsymbol{FQ}) - 2\operatorname{tr}(\boldsymbol{FQQ}^T\boldsymbol{F}^T) + \operatorname{tr}(\boldsymbol{I})\right) =$$

$$\frac{\lambda}{8}\left(\operatorname{tr}(\boldsymbol{FF}^T) - \operatorname{tr}(\boldsymbol{I})\right)^2 + \frac{\mu}{4}\left(\operatorname{tr}((\boldsymbol{F}^T\boldsymbol{FQ})^T\boldsymbol{F}^T\boldsymbol{FQ}) - 2\operatorname{tr}(\boldsymbol{FF}^T) + \operatorname{tr}(\boldsymbol{I})\right) =$$

$$\frac{\lambda}{8}\left(\operatorname{tr}(\boldsymbol{FF}^T) - \operatorname{tr}(\boldsymbol{I})\right)^2 + \frac{\mu}{4}\left(\operatorname{tr}(\boldsymbol{F}^T\boldsymbol{FQ}(\boldsymbol{F}^T\boldsymbol{FQ})^T) - 2\operatorname{tr}(\boldsymbol{FF}^T) + \operatorname{tr}(\boldsymbol{I})\right) =$$

$$\frac{\lambda}{8}\left(\operatorname{tr}(\boldsymbol{FF}^T) - \operatorname{tr}(\boldsymbol{I})\right)^2 + \frac{\mu}{4}\left(\operatorname{tr}(\boldsymbol{F}^T\boldsymbol{FQQ}^T\boldsymbol{F}^T\boldsymbol{F}) - 2\operatorname{tr}(\boldsymbol{FF}^T) + \operatorname{tr}(\boldsymbol{I})\right) =$$

$$\frac{\lambda}{8}\left(\operatorname{tr}(\boldsymbol{FF}^T) - \operatorname{tr}(\boldsymbol{I})\right)^2 + \frac{\mu}{4}\left(\operatorname{tr}(\boldsymbol{F}^T\boldsymbol{FF}^T\boldsymbol{F}) - 2\operatorname{tr}(\boldsymbol{FF}^T) + \operatorname{tr}(\boldsymbol{I})\right) =$$

$$\frac{\lambda}{2}\left(\frac{1}{2}\operatorname{tr}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})\right)^2 + \mu\operatorname{tr}\left(\frac{1}{4}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})^2\right) = W(\boldsymbol{F})$$

So we have proved that the model is isotropic.

3. Derive the second Piola-Kirchhoff stress $\boldsymbol{S}$.

The second Piola Kirchhoff stress is calculated as follows:

$$S_{IJ} = \frac{\partial W}{\partial E_{IJ}} = \frac{\partial}{\partial E_{IJ}}\left(\frac{\lambda}{2}(E_{II})^2 + \mu E_{JK}E_{JK}\right) = \lambda E_{II}\delta_{IJ} + 2\mu E_{JK}\delta_{IJ}\delta_{JK} = \lambda E_{II}\delta_{IJ} + 2\mu E_{IJ}$$

$$\boldsymbol{S} = \lambda\operatorname{tr}(\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E}$$

4. For a uniform deformation of a rod aligned with the X axis ($x = \Lambda X, y = Y, z = Z$, where $\Lambda > 0$ is the stretch ratio along the X direction) derive the relation between the nominal stress P (the xX component of the first Piola-Kirchhoff stress) and the stretch ratio $\Lambda, P(\Lambda)$, and plot it.

The deformation map is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \Lambda X \\ Y \\ Z \end{bmatrix}$$

This implies:

$$\boldsymbol{F} = \frac{\partial\varphi}{\partial\boldsymbol{X}} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \begin{bmatrix} \Lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \begin{bmatrix} \frac{1}{2}(\Lambda^2 - 1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{S} = \lambda\,\mathrm{tr}(\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E} = \begin{bmatrix} (\Lambda^2 - 1)\left(\frac{\lambda}{2} + \mu\right) & 0 & 0 \\ 0 & \frac{\lambda}{2}(\Lambda^2 - 1) & 0 \\ 0 & 0 & \frac{\lambda}{2}(\Lambda^2 - 1) \end{bmatrix}$$

$$\boldsymbol{P} = \boldsymbol{F}\boldsymbol{S} = \begin{bmatrix} \Lambda(\Lambda^2 - 1)\left(\frac{\lambda}{2} + \mu\right) & 0 & 0 \\ 0 & \frac{\lambda}{2}(\Lambda^2 - 1) & 0 \\ 0 & 0 & \frac{\lambda}{2}(\Lambda^2 - 1) \end{bmatrix}$$

$$P_{xX} = \Lambda(\Lambda^2 - 1)\left(\frac{\lambda}{2} + \mu\right)$$

This last result has been plotted:

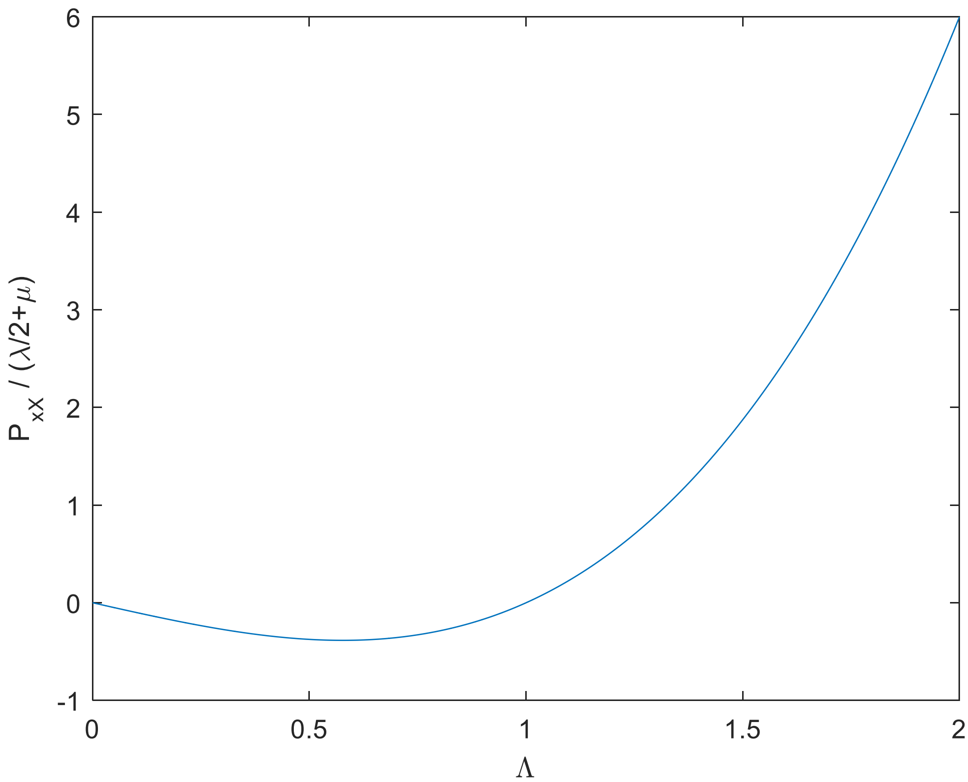

Figure 1: Stress P_{xX} vs Λ

5. Is the relation $P(\Lambda)$ monotonic? If not, derive the critical stretch Λ_{crit} at which the model fails with zero stiffness. Does this critical stretch depend on the elastic constants? Show that the material does not satisfy the growth conditions

$$W(\boldsymbol{E}) \longrightarrow +\infty \text{ when } J \longrightarrow 0^+.$$

From the previous plot it is seen that the relation is not monotonic:

$$\frac{\partial P_{xX}}{\partial \Lambda}(\Lambda_{crit}) = \left(\frac{\lambda}{2}+\mu\right)(3\Lambda^2-1) = 0 \rightarrow 3\Lambda_{crit}^2 - 1 = 0 \rightarrow \Lambda_{crit} = \sqrt{\frac{1}{3}}$$

This is physically inconsistent as this means that over some point, the material However, the value of Λ_{crit} is independent of the elastic constants.

The determinant of the deformation is:

$$J = |\boldsymbol{F}| = \Lambda$$

It is trivial to show that:

$$\lim_{\Lambda\to 0} W(\boldsymbol{E}) = 0$$

So the growth condition is not satisfied. This means that the material could be compressed to the limit of occupying a null volume with a finite amount of energy and that is physically inconsistent.

6. Consider now the modified Kirchhoff Saint-Venant material model:

$$W(\boldsymbol{E}) = \frac{\lambda}{2}(\ln J)^2 + \mu\,\mathrm{tr}(\boldsymbol{E}^2)$$

Does this model circumvent the drawbacks of the previous model?

To compute the second Piola-Kirchhoff stress tensor first, the derivative $\partial J/\partial \boldsymbol{E}$ is computed:

$$\frac{\partial J}{\partial E_{IJ}} = \frac{\partial |F|}{\partial F_{kL}}\frac{\partial F_{kL}}{\partial E_{IJ}} = |F| F_{kL}^{-T}\left(\frac{\partial E_{IJ}}{\partial F_{kL}}\right)^{-1} = J F_{kL}^{-T}\left(\frac{1}{2}\frac{\partial\left(F_{mI}F_{mJ}-\delta_{IJ}\right)}{\partial F_{kL}}\right)^{-1} =$$

$$2JF_{kL}^{-T}\left(F_{mJ}\delta_{mk}\delta_{IL} + F_{mI}\delta_{mk}\delta_{JL}\right)^{-1} = 2JF_{kL}^{-T}\left(F_{kJ}\delta_{IJ} + F_{kI}\delta_{JL}\right)^{-1} =$$

$$2J\left(F_{kJ}F_{kI} + F_{kI}F_{kJ}\right)^{-1} = 2J\left(2F_{kI}F_{kJ}\right)^{-1} = J\left(F_{kI}F_{kJ}\right)^{-1}$$

$$\frac{\partial J}{\partial \boldsymbol{E}} = J(\boldsymbol{F}^T\boldsymbol{F}) = J\boldsymbol{C} = J(2\boldsymbol{E}+\boldsymbol{I})$$

This means that the second Piola-Kirchhoff stress tensor is:

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{E}} = \frac{\lambda}{2}\cdot 2\ln(J)\frac{\partial J}{\partial \boldsymbol{E}} + 2\mu\boldsymbol{E} = \lambda\ln J\cdot J(2\boldsymbol{E}+\boldsymbol{I})^{-1} + 2\mu\boldsymbol{E}$$

For the deformation case of the last section:

$$S_{xX} = \lambda(\ln\Lambda)\Lambda(\Lambda^2)^{-1} + \mu(\Lambda^2 - 1), \qquad P_{xX} = \Lambda\big(\lambda(\ln\Lambda)\Lambda^{-1} + \mu(\Lambda^2 - 1)\big) = \lambda\ln\Lambda + \mu(\Lambda^3 - \Lambda)$$

In this case, it can be seen that the stress is monotonic under the condition that $\frac{\mu}{\lambda} < \frac{9}{2}$

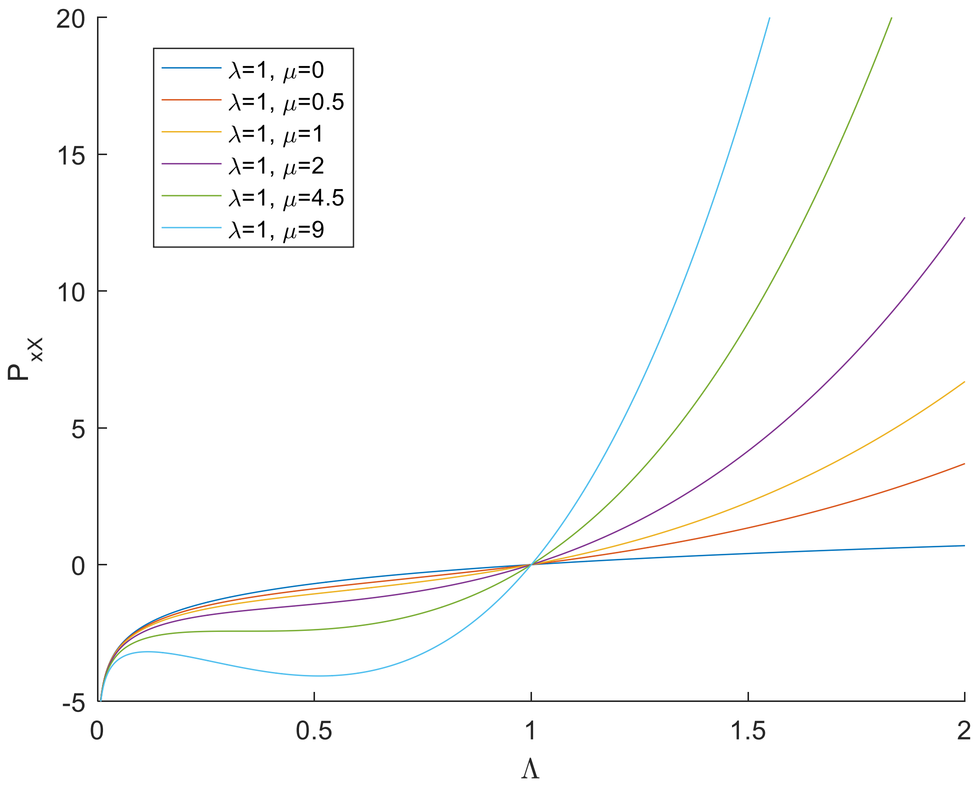

Figure 2: Stress P_{xX} vs Λ

In any case, the logarithmic term guarantees the growth condition as

$$\lim_{\Lambda\to 0} W(\boldsymbol{E}) = -\infty$$

So, we can conclude that this model is physical consistent given the condition $\frac{\mu}{\lambda} < \frac{9}{2}$ so it circumvents the drawbacks of the previous one.