Universitat Politècnica de Catalunya

Computational Solid Mechanics

Master's Degree in Numerical Methods in Engineering

Coupled Problems Homeworks

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1 Transmission conditions

1.1 Deflection of an Euler-Bernoulli beam

1.1.a Space of functions where both v and δv belong.

Let us define for $k = 1, 2, ...$

$$H^k(\Omega) = \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega), |\alpha| \leq k\} \tag{1}$$

The Sobolev space $H^k(\Omega)$ consists of all functions v on Ω, that, together with its partial derivatives of order α, belong to the Hilbert space $L^2(\Omega)$. Let us define this space of square integrable functions as

$$L^2(\Omega) = \{v : v \quad \textit{is defined on} \quad \Omega \quad \textit{and} \int_\Omega v^2 dx < \infty\} \tag{2}$$

Now let us introduce the space in which δv is

$$H_0^2(\Omega) = \{u \in H^2(\Omega) : u = \frac{\delta u}{\delta x} = 0 \quad on \quad \Gamma\} \tag{3}$$

So it is clear that it is in this space since the boundary conditions are the same. As for v, the restriction is lower, thus is belongs to $H^2(\Omega)$, as its first and second derivatives need to be square integrable, as well as the function v itself.

1.1.b Obtain the transmission conditions at P implied by regularity requirements

It is clear that we will need to satisfy V_h corresponding to the fourth order boundary value problem. The finite element space V_h will belong to $H^2(\Omega)$ if and only if the functions $v \in V_h$ and their first derivatives are continuous, otherwise the derivative does not exist as a function in $L^2(\Omega)$. In order for v to not be discontinuous across the the element side the transmission conditions read

$$\begin{cases} [\![v]\!] = v(P^+) - v(P^-) = 0 \\ [\![\frac{\partial v}{\partial x}]\!] = 0 \end{cases} \tag{4}$$

1.1.c Transmission conditions at P that follow by imposing in the PTV that the integral is additive

Now we will use the starting equation and integrate it by parts considering that now $[0, L] = [0, P] \cup (P, L]$. That is,

$$\underbrace{EI \int_0^P \delta v \frac{d^4 v}{dx^4}}_{\text{A}} + \underbrace{EI \int_P^L \delta v \frac{d^4 v}{dx^4}}_{\text{B}} = \int_0^P \delta v f + \int_P^L \delta v f = \int_0^L \delta v f \tag{5}$$

Now, integrating by parts terms A and B,

$$\begin{aligned} A \longrightarrow EI \int_0^P \delta v \frac{d^4 v}{dx^4} &= EI \left(\int_0^P \frac{d\delta v}{dx} \frac{d^3 v}{dx^3} dx + \left[\delta v \frac{d^3 v}{dx^3} \right]_0^P \right) = \\ &= EI \left(\int_0^P \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} dx + \left[\delta v \frac{d^3 v}{dx^3} \right]_0^P + \left[\frac{d\delta v}{dx} \frac{d^2 v}{dx^2} \right]_0^P \right) \end{aligned} \tag{6}$$

$$\begin{aligned} B \longrightarrow EI \int_P^L \delta v \frac{d^4 v}{dx^4} &= EI \left(\int_P^L \frac{d\delta v}{dx} \frac{d^3 v}{dx^3} dx + \left[\delta v \frac{d^3 v}{dx^3} \right]_P^L \right) = \\ &= EI \left(\int_P^L \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} dx + \left[\delta v \frac{d^3 v}{dx^3} \right]_P^L + \left[\frac{d\delta v}{dx} \frac{d^2 v}{dx^2} \right]_P^L \right) \end{aligned} \tag{7}$$

Now it is clear that in order to reach the final term the sum of the following is equal to zero,

$$\begin{aligned} \left[\delta v \frac{d^3 v}{dx^3} \right]_0^P + \left[\frac{d\delta v}{dx} \frac{d^2 v}{dx^2} \right]_0^P + \left[\delta v \frac{d^3 v}{dx^3} \right]_P^L + \left[\frac{d\delta v}{dx} \frac{d^2 v}{dx^2} \right]_P^L &= 0 \\ \delta v \frac{d^3 v}{dx^3} \bigg|_{x=P^-} + \frac{d\delta v}{dx} \frac{d^2 v}{dx^2} \bigg|_{x=P^-} - \delta v \frac{d^3 v}{dx^3} \bigg|_{x=P^+} - \frac{d\delta v}{dx} \frac{d^2 v}{dx^2} \bigg|_{x=P^+} &= 0 \end{aligned} \tag{8}$$

In order for the latter to be true, given that the function δv is arbirtary, the following has to be true

$$[\![\delta v]\!] = [\![\frac{d\delta v}{dx}]\!] = [\![\frac{d^2 v}{dx^2}]\!] = [\![\frac{d^3 v}{dx^3}]\!] = 0 \tag{9}$$

1.2 The Maxwell problem

1.2.a Write a variational statement of the problem. Postulate the space of functions where u must belong.

There are two identities that will be used to simplify and get rid of the curl of the curl.

- The cross product rule $\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B})$.
- The divergence theorem $\int_\Omega \boldsymbol{\nabla} \cdot F d\Omega = \int_{\partial\Omega} \mathbf{F} \cdot n d\Gamma$.

When writing the variational formulation, a test function $\mathbf{v}$ will multiply the equation and then it will be integrated on the domain.

$$\int_\Omega \mathbf{v} \cdot (\nu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \tag{10}$$

When using the first identity mentioned, considering that $\mathbf{A} = \boldsymbol{\nabla} \times u$ and $\mathbf{B} = \mathbf{v}$

$$\int_\Omega \mathbf{v} \cdot (\nu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u}) = \int_\Omega (\nu \boldsymbol{\nabla} \times u) \cdot (\boldsymbol{\nabla} \times \mathbf{v}) + \underbrace{\int_\Omega \boldsymbol{\nabla} \cdot (\mathbf{v} \times (\nu \boldsymbol{\nabla} \times \mathbf{u}))}_{\text{C}} \tag{11}$$

Now by applying the divergence theorem on C, it is reached

$$\int_\Omega \mathbf{v} \cdot (\nu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u}) = \int_\Omega (\nu \boldsymbol{\nabla} \times u) \cdot (\boldsymbol{\nabla} \times \mathbf{v}) + \int_{\partial\Omega} (\mathbf{v} \times (\nu \boldsymbol{\nabla} \times \mathbf{u})) \cdot \mathbf{n} \tag{12}$$

If considering that $\mathbf{n} \times (\boldsymbol{\nabla} \times u \times \mathbf{v}) = -\mathbf{v} \times (\boldsymbol{\nabla} \times u \cdot \mathbf{n})$, it is possible to rewrite the boundary term and reach the fact that it vanishes due to $\mathbf{n} \times \mathbf{u} = 0$. Hence the variational formulation is

$$\int_\Omega \mathbf{f} \cdot \mathbf{v} = \int_\Omega (\nu \boldsymbol{\nabla} \times u) \cdot (\boldsymbol{\nabla} \times \mathbf{v}) \tag{13}$$

As the left-hand side term is in $L^1(\Omega)$, so is the right-hand side. That means that both cross products need to be in $L^2(\Omega)$. Therefore we can write as in the last exercise two different Hilbert spaces, in this case $H_0^0(\Omega)$ and $H^0(\Omega)$ with the difference that now the conditions are applied on the curl, meaning that

$$H^0(curl, \Omega) = \{u \in L^2(\Omega) : \boldsymbol{\nabla} \times \mathbf{u} \in L^2(\Omega)\} \tag{14}$$

$$H_0^0(curl, \Omega) = \{u \in L^2(\Omega) : \mathbf{u} \times \mathbf{n} = 0\} \tag{15}$$

Then it is clear that $\mathbf{u}$ and $\mathbf{v}$ belong to $H_0^0(curl, \Omega)$.

1.2.b Obtain the transmission conditions across Γ implied by regularity requirements

For any orientable and closed surface Γ the unit normal vector $\mathbf{n}$ on Γ is outwardly oriented from the interior domain enclosed by Γ towards the outer domain. Let $\mathbf{n}$ be a vector fielf on Γ, then we denote by $\mathbf{v}_T = \mathbf{n} \times (\mathbf{v} \times \mathbf{n})$ the vector field of its tangent components and the space of $L^2(\Omega)$ in which it belongs with

$$L_t^2(\Gamma) = \{u \in L^2(\Gamma) : \mathbf{u} \cdot \mathbf{n} = 0 \quad on \quad \Gamma\} \tag{16}$$

Then, for a vector field $\mathbf{u} \in H(curl, \Omega)$ the transmission condition is that the jump $[\![\mathbf{u} \times \mathbf{n}]\!]_\Gamma = 0$, or the tangential velocity on the surface that intersects.

1.2.c Obtain the transmission conditions across Γ that follow by imposing in the variational form of the problem that the integral is additive

It was shown in the first part that the variational form on the domain had to form of (12). Later we saw that the contribution on the boundary vanished due to $\mathbf{n} \times \mathbf{u} = 0$ on the boundary. Now there is another boundary Γ which has to be considered in the equation. So, when the integral is additive, there are two subdomains whose contributions are

$$\begin{aligned}\int_\Omega \mathbf{v} \cdot (\nu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u}) = \int_{\Omega_1} (\nu \boldsymbol{\nabla} \times u) \cdot (\boldsymbol{\nabla} \times \mathbf{v}) + \int_\Gamma (\mathbf{v} \times (\nu \boldsymbol{\nabla} \times \mathbf{u})) \cdot \mathbf{n}|_{\Omega_1} + \\ + \int_{\Omega_2} (\nu \boldsymbol{\nabla} \times u) \cdot (\boldsymbol{\nabla} \times \mathbf{v}) + \int_\Gamma (\mathbf{v} \times (\nu \boldsymbol{\nabla} \times \mathbf{u})) \cdot \mathbf{n}|_{\Omega_2}\end{aligned} \tag{17}$$

In order for the formulation to be consistent with what was found in (13), the sum of the contributions on the boundary must vanish, so that

$$\int_\Gamma (\mathbf{v} \times (\nu \boldsymbol{\nabla} \times \mathbf{u})) \cdot \mathbf{n}|_{\Omega_1} + \int_\Gamma (\mathbf{v} \times (\nu \boldsymbol{\nabla} \times \mathbf{u})) \cdot \mathbf{n}|_{\Omega_2} = 0 \tag{18}$$

So, as we saw, now the jump of the term of the projection of the curl on the normal to the surface Γ must be zero, meaning that the transmission condition is

$$[\![(\nu \boldsymbol{\nabla} \times \mathbf{u})) \cdot \mathbf{n}]\!] = 0 \tag{19}$$

1.3 The Navier equations

1.3.a Variational form of the equations

The first equation is written in the form

$$\underbrace{-2\mu \int_\Omega \mathbf{v} \cdot (\boldsymbol{\nabla} \cdot \nabla^S \mathbf{u})}_{\text{A}} \underbrace{-\lambda \int_\Omega \mathbf{v} \cdot (\nabla(\boldsymbol{\nabla} \cdot \mathbf{u}))}_{\text{B}} = \rho \int_\Omega \mathbf{v} \cdot \mathbf{b} \tag{20}$$

Now, term A will simply be integrated by parts, but term B will require the employment of the chain rule and the divergence theorem to be simplified.

$$A \longrightarrow -2\mu \int_\Omega \mathbf{v} \cdot (\boldsymbol{\nabla} \cdot \nabla^S \mathbf{u}) = 2\mu \int_\Omega \nabla \mathbf{v} : \nabla^S \mathbf{u} - 2\mu \int_{\partial\Omega} \mathbf{v} \cdot \nabla^S \mathbf{u} \cdot \mathbf{n} \tag{21}$$

$$B \longrightarrow -\lambda \int_\Omega \mathbf{v} \cdot (\nabla(\boldsymbol{\nabla} \cdot \mathbf{u})) = -\underbrace{\int_\Omega \boldsymbol{\nabla} \cdot (\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v}))}_{\text{C}} + \underbrace{\int_\Omega (\boldsymbol{\nabla} \cdot v)(\boldsymbol{\nabla} \cdot u)}_{\text{D}} \tag{22}$$

Then, (20) eventually is

$$2\mu \int_\Omega \nabla \mathbf{v} : \nabla^S \mathbf{u} - 2\mu \int_{\partial\Omega} \mathbf{v} \cdot \nabla^S \mathbf{u} \cdot \mathbf{n} + \lambda \int_\Omega (\boldsymbol{\nabla} \cdot \mathbf{v})(\boldsymbol{\nabla} \cdot \mathbf{v}) - \lambda \underbrace{\int_{\partial\Omega} (\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v})) \cdot \mathbf{n}}_{\text{C}} = \rho \int_\Omega \mathbf{v} \cdot \mathbf{b} \tag{23}$$

As for the second equation, which reads

$$-\mu \int_\Omega \mathbf{v} \cdot \Delta \mathbf{u} - (\lambda + \mu) \int_\Omega \mathbf{v} \cdot \nabla(\boldsymbol{\nabla} \cdot \mathbf{u}) = \rho \int_\Omega \mathbf{v} \cdot \mathbf{b} \tag{24}$$

The first term in the right-hand side will be integrated by parts and the second term will be treated exactly as term B in the previous equation, yielding

$$\mu \int_\Omega \nabla \mathbf{v} : \nabla \mathbf{u} - \mu \int_{\partial\Omega} \mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{n} + (\lambda + \mu) \int_\Omega (\boldsymbol{\nabla} \cdot \mathbf{v})(\boldsymbol{\nabla} \cdot \mathbf{v}) - (\lambda + \mu) \int_{\partial\Omega} (\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v})) \cdot \mathbf{n} = \rho \int_\Omega \mathbf{v} \cdot \mathbf{b} \tag{25}$$

As for the third equation, which reads

$$\mu \int_\Omega \mathbf{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u}) - (\lambda + 2\mu) \int_\Omega \mathbf{v} \cdot \nabla(\boldsymbol{\nabla} \cdot \mathbf{u}) = \rho \int_\Omega \mathbf{v} \cdot \mathbf{b} \tag{26}$$

The first term in the right-hand side will be integrated according to the expression that was seen in the previous exercises and the second term will be treated exactly as term B in the first equation.

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u} = \nabla(\boldsymbol{\nabla} \cdot \mathbf{u}) - \nabla^2 \mathbf{u} \tag{27}$$

Therefore, if we introduce this in the previous equation we have

$$\begin{aligned} \mu \int_\Omega \mathbf{v} \cdot (\nabla(\boldsymbol{\nabla} \cdot \mathbf{u}) - \nabla^2 \mathbf{u}) - (\lambda + 2\mu) \int_\Omega \mathbf{v} \cdot \nabla(\boldsymbol{\nabla} \cdot \mathbf{u}) = \rho \int_\Omega \mathbf{v} \cdot \mathbf{b} \\ \mu \int_\Omega \mathbf{v} \cdot (\nabla(\boldsymbol{\nabla} \cdot \mathbf{u}) - \nabla^2 \mathbf{u}) - (\lambda + \mu) \int_\Omega \mathbf{v} \cdot \nabla(\boldsymbol{\nabla} \cdot \mathbf{u}) - \mu \int_\Omega \mathbf{v} \cdot \nabla(\boldsymbol{\nabla} \cdot \mathbf{u}) = \rho \int_\Omega \mathbf{v} \cdot \mathbf{b} \\ -\mu \int_\Omega \mathbf{v} \cdot (\nabla^2 \mathbf{u}) - (\lambda + \mu) \int_\Omega \mathbf{v} \cdot \nabla(\boldsymbol{\nabla} \cdot \mathbf{u}) = \rho \int_\Omega \mathbf{v} \cdot \mathbf{b} \end{aligned} \tag{28}$$

We have then reach the same as in the second expression, as they are equivalent expressions. Therefore the variational form will be the same.

Now it is clear that v will belong to $H^1(\Omega)$ and u will belong to $H_0^1(\Omega)$.

1.3.b Transmission conditions across Γ

Regarding the additive properties of the integrals, it has been shown that, when integrating across a domain with a surface Γ intersected, the contributions of the terms along the part of the boundary of each sub-domain corresponding to Γ must sum zero in order to recover the weak form. In the case of the particular problem of the equations two and three, this translates into

$$\begin{aligned} \int_\Gamma \mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{n}|_{\Omega_1} + \int_\Gamma \mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{n}|_{\Omega_2} = 0 \\ \int_\Gamma (\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v})) \cdot \mathbf{n}|_{\Omega_1} + \int_\Gamma (\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v})) \cdot \mathbf{n}|_{\Omega_2} = 0 \end{aligned} \tag{29}$$

Now the jump of the term of the projection of the gradient on the normal to the surface Γ must be zero, as well as the projection of the divergence, meaning that the transmission conditions are

$$[\![(\nabla \mathbf{u})) \cdot \mathbf{n}]\!] = 0 \qquad [\![(\boldsymbol{\nabla} \cdot \mathbf{u}) \cdot \mathbf{n}]\!] = 0 \tag{30}$$

Clearly the functions cannot be discontinuous across the surface, otherwise they cannot belong to $H^1(\Omega)$.

2 Domain decomposition methods

2.1 Problem 1

2.1.a Write down an i-b-s scheme based on a Schwarz additive domain decomposition method

The additive Schwarz method is a parallel Jacobi-like scheme in which the values at the boundary of the sub-domain are calculated by the previous iteration, that is, $l = k - 1$. As there are two domains that overlap each other, we define

$$\begin{cases} \Omega_1 = \{x \in [0, L_1]\}, \quad \Gamma_{12} = L_1 \\ \Omega_2 = \{x \in [L_2, L]\}, \quad \Gamma_{21} = L_2 \end{cases} \tag{31}$$

With this, we will consider for easiness that the beam is clamped at both ends. We will use the differential operator as follows as well

$$EI\frac{d^4v}{dx^4} = f \longrightarrow EI\mathscr{L}v = f \tag{32}$$

As for the first sub-domain, it is only necessary to recall the transmission conditions.

$$\begin{cases} EI\mathscr{L}v_1^k = f \quad in \quad \Omega_1 \\ v_1^k = 0 \quad on \quad x = 0 \\ \frac{dv_1^k}{dx} = 0 \quad on \quad x = 0 \\ v_1^k = v_2^{k-1} \quad on \quad \Gamma_{12} \\ \frac{dv_1^k}{dx} = \frac{dv_2^{k-1}}{dx} \quad on \quad \Gamma_{12} \end{cases} \tag{33}$$

As for subdomain 2,

$$\begin{cases} EI\mathscr{L}v_2^k = f \quad in \quad \Omega_2 \\ v_2^k = 0 \quad on \quad x = L \\ \frac{dv_2^k}{dx} = 0 \quad on \quad x = L \\ v_2^k = v_1^{k-1} \quad on \quad \Gamma_{21} \\ \frac{dv_2^k}{dx} = \frac{dv_1^{k-1}}{dx} \quad on \quad \Gamma_{21} \end{cases} \tag{34}$$

2.1.b Obtain the matrix version of the previous scheme once space has been discretized using finite elements

When discretizing the space, we define the weak form as was done in the first exercise with a bilinear form for the matrix and a linear form for the right-hand side term, obtaining

$$a(\mathbf{v},\mathbf{u}) = \int_{\Omega} \nabla\mathbf{v} : \nabla\mathbf{u}, \quad \mathbf{v} \in H_0^2(\Omega), \quad \mathbf{u} \in H^2(\Omega) \tag{35}$$

$$h(\mathbf{v},\mathbf{u}) = \int_{\Omega} \mathbf{v} f, \quad \mathbf{v} \in H_0^2(\Omega) \tag{36}$$

After this is done, the matrix notation for an iterative scheme is used to obtain for the first subdomain,

$$\begin{aligned} A_{11}v_1^k &= f_1 - A_{1\Gamma}v_2^{k-1}(L_1) \\ A_{22}v_2^k &= f_2 - A_{2\Gamma}v_1^{k-1}(L_2) \end{aligned} \tag{37}$$

Where $A_{11} = a1(N_i, N_j), A_{22} = a2(N_i, N_j), f_1 = h1(N_i), f_2 = h2(N_i)$.

The values at the interfaces can be calculated easily by the values at the other subdomain at the previous iteration.

2.2 Problem 2

2.2.a Write down an i-b-s scheme based on the Dirichlet-Neumann coupling

As the transmission conditions for this exercise cannot be imposed at the same time, it is necessary to iterate by sub-domains. For this we define $\Omega = \Omega_1 \cup \Omega_2, \Gamma_{12} = \Omega_1 \cap \Omega_2), \Gamma_i = \partial\Omega_i \cap \partial\Omega$. Now, if recalling the transmission conditions that was found, $[\![(\nu\boldsymbol{\nabla} \times \mathbf{u})) \cdot \mathbf{n}]\!] = 0$, and the boundary conditions for that specific problem, the iterations are for each sub-domain are:

$$\begin{cases} \nu\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u_1^k} = \mathbf{f} \quad in \quad \Omega_1 \\ \mathbf{n} \times \mathbf{u_1^k} = 0 \quad on \quad \Gamma_1 \\ \nu(\boldsymbol{\nabla} \times \mathbf{u_1^k}) \cdot \mathbf{n} = \nu(\boldsymbol{\nabla} \times \mathbf{u_2^{k-1}}) \cdot \mathbf{n} \quad on \quad \Gamma_{12} \end{cases} \tag{38}$$

As for subdomain 2,

$$\begin{cases} \nu\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u_2^k} = \mathbf{f} \quad in \quad \Omega_2 \\ \mathbf{n} \times \mathbf{u_2^k} = 0 \quad on \quad \Gamma_2 \\ \mathbf{n} \times \mathbf{u_2^k} = \mathbf{n} \times \mathbf{u_1^l} \quad on \quad \Gamma_{21} \end{cases} \tag{39}$$

Here it is seen that sub-domain 1 solves for Neumann and sub-domain 2 solves for Dirichlet. If $l = k - 1$ we have a Jacobi scheme (in parallel) and if $l = k$ we have a Gauss-Seidel scheme (sequential).

2.2.b Obtain the expression of the Steklov-Poincare operator of the problem.

The problem consists of finding $\varphi \in H^{1/2}(\Gamma_{12})$ such that $\mathscr{S}\phi = \mathscr{G}$. For that, the problem solution will be split in two parts, so that the first part will satisfy the homogeneous boundary conditions and the other will have an unspecified value φ at the interface between two sub-domains so that in each sub-domain we have

$$\begin{cases} \nu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u_i^0} = \mathbf{f} \quad in \quad \Omega_i \\ \mathbf{n} \times \mathbf{u_i^0} = 0 \quad on \quad \Gamma_1 \\ \mathbf{n} \times \mathbf{u_i^0} = 0 \quad on \quad \Gamma_{12} \end{cases} \tag{40}$$

$$\begin{cases} \nu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \tilde{\mathbf{u_i}} = \mathbf{f} \quad in \quad \Omega_i \\ \mathbf{n} \times \tilde{\mathbf{u_i}} = 0 \quad on \quad \Gamma_1 \\ \mathbf{n} \times \tilde{\mathbf{u_i}} = \varphi \quad on \quad \Gamma_{12} \end{cases} \tag{41}$$

Again, recalling the transmission conditions, we can define the previous operators as

$$\begin{gathered} \mathscr{S} : H^{1/2}(\Gamma_{12}) \longrightarrow H^{-1/2}(\Gamma_{12}) \\ \varphi \longrightarrow \nu_1(\boldsymbol{\nabla} \times \tilde{\mathbf{u_1}}) \cdot \mathbf{n} - \nu_2(\boldsymbol{\nabla} \times \tilde{\mathbf{u_2}}) \cdot \mathbf{n} \\ \mathscr{G} = -\nu_1(\boldsymbol{\nabla} \times \mathbf{u_1^0}) \cdot \mathbf{n} + \nu_2(\boldsymbol{\nabla} \times \mathbf{u_2^0}) \cdot \mathbf{n} \end{gathered} \tag{42}$$

2.2.c Obtain the matrix version of the previous scheme once space has been discretized using finite elements

For a Neumann-Dirichlet iterative shcheme we have

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A^1_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} U_1^k \\ U_\Gamma^k \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2} U_2^{k-1} - A^2_{\Gamma\Gamma} U_\Gamma^{k-1} \end{bmatrix} \tag{43}$$

And the Dirichlet conditions are satisfied through $A_{22}U_2^k = F_2 - A_{2\Gamma}U_\Gamma^l$. l can be as usual k or $k-1$.

2.3 Problem 3

2.3.a Write down an i-b-s scheme based on the Robin-Dirichlet coupling

Let us consider a Jacobi scheme. The first sub-domain will be solved by enforcing the transmission condition at the interface with a Dirichlet coupling and with a Robin coupling on the second sub-domain such that

$$\begin{cases} k_1 \Delta \mathbf{u_1^k} = \mathbf{f} \quad in \quad \Omega_1 \\ \mathbf{u_1^k} = 0 \quad on \quad \Gamma_1 \\ \mathbf{u_1^k} = \mathbf{u_2^{k-1}} \quad on \quad \Gamma_{12} \end{cases} \tag{44}$$

$$\begin{cases} k_2 \Delta \mathbf{u_2^k} = \mathbf{f} \quad in \quad \Omega_2 \\ \mathbf{u_2^k} = 0 \quad on \quad \Gamma_2 \\ k_2 \frac{\partial \mathbf{u_2^k}}{\partial n} + \gamma_2 \mathbf{u_2^k} = k_1 \frac{\partial \mathbf{u_1^{k-1}}}{\partial n} + \gamma_2 \mathbf{u_1^{k-1}} \quad on \quad \Gamma_{12} \end{cases} \tag{45}$$

2.3.b Obtain the matrix version of the previous scheme once space has been discretized using finite elements

The iterative matrix version will be the same as the Neumann-Dirichlet system but taking into account the Robin condition this time. For sub-domain 2 we have simply

$$A_{22} U_2^k = F_2 - A_{2\Gamma} U_\Gamma^{k-1} \tag{46}$$

And for sub-domain 1 we have a matrix system of equations with the vector unknowns U_1^k at the domain and the vector unknowns U_Γ^k at the boundary Γ_{12}. Taking into account that the Robin condition acts on the boundary Γ_{12}, the variable γ can therefore only affect the vector U_Γ^k as

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^1 + \gamma_1 \mathbf{I} \end{bmatrix} \begin{bmatrix} U_1^k \\ U_\Gamma^k \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2} U_2^{k-1} - (A_{\Gamma\Gamma}^2 - \gamma_1 \mathbf{I}) U_\Gamma^{k-1} \end{bmatrix} \tag{47}$$

2.3.c Obtain the Schur complement as discrete version of the Steklov-Poincare operator

To do this, we need to substitute equation (46) into (47) to obtain the system of equations $\mathbf{S U_\Gamma} = \mathbf{G}$.

$$\begin{cases} U_1 = A_{11}^{-1}(F_1 - A_{1\Gamma} U_\Gamma) \\ U_2 = A_{22}^{-1}(F_2 - A_{2\Gamma} U_\Gamma) \end{cases} \tag{48}$$

Now we need to introduce this into

$$A_{\Gamma 1}U_1 + (A_\Gamma \Gamma^1 + \gamma_1 \mathbf{I})U_\Gamma = F_\Gamma - A_{\Gamma 2}U_2 - (A^2_{\Gamma\Gamma} - \gamma_1 \mathbf{I})U_\Gamma \tag{49}$$

Which gives the following matrices

$$\begin{aligned} \mathbf{S} &= A^1_{\Gamma\Gamma} + \gamma_1 \mathbf{I} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma} \\ \mathbf{G} &= F_\Gamma - A_{\Gamma 1}A_{11}^{-1}F_1 - A_{\Gamma 2}A_{22}^{-1}F_2 - (A_\Gamma \Gamma^2 + \gamma_1 \mathbf{I})U_\Gamma \end{aligned} \tag{50}$$

2.3.d Identify the preconditioner for the Schur complement equation arising from the iterative scheme of section (a)

A Richardson iterative scheme to solve this problem with preconditioner would be

$$U_\Gamma^k = U_\Gamma^{k-1} + P^{-1}(G - SU_\Gamma^{k-1}) \tag{51}$$

Let us consider this time the Gauss-Seidel i-b-s method

$$\begin{cases} U_1^k = A_{11}^{-1}(F_1 - A_{1\Gamma}U_\Gamma)^k \\ U_2^{k-1} = A_{22}^{-1}(F_2 - A_{2\Gamma}U_\Gamma)^{k-1} \end{cases} \tag{52}$$

Putting this into the iterative equation that we have seen already

$$A_{\Gamma 1}U_1^k + (A_\Gamma \Gamma^1 + \gamma_1 \mathbf{I})U_\Gamma^k = F_\Gamma - A_{\Gamma 2}U_2^{k-1} - (A^{(2)}_{\Gamma\Gamma} - \gamma_1 \mathbf{I})U_\Gamma^{k-1} \tag{53}$$

will give us the equation we seek. We have that

$$A_{\Gamma 1}(A_{11}^{-1}(F_1 - A_{1\Gamma}U_\Gamma^k)) + (A_\Gamma \Gamma^1 + \gamma_1 \mathbf{I})U_\Gamma^k = F_\Gamma - A_{\Gamma 2}(A_{22}^{-1}(F_2 - A_{2\Gamma}U_\Gamma^{k-1})) - (A^{(2)}_{\Gamma\Gamma} - \gamma_1 \mathbf{I})U_\Gamma^{k-1} \tag{54}$$

Now, in order to simplify this, we need to group terms by dividing the Schur component into two terms, namely $S = S_1 + S_2$. Now,

$$\begin{aligned} S_1 &= A^1_{\Gamma\Gamma} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} \\ S_2 &= A^2_{\Gamma\Gamma} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma} \end{aligned} \tag{55}$$

And the matrix G was computed before and was

$$\mathbf{G} = F_\Gamma - A_{\Gamma 1}A_{11}^{-1}F_1 - A_{\Gamma 2}A_{22}^{-1}F_2 - (A_\Gamma \Gamma^2 + \gamma_1 \mathbf{I})U_\Gamma^{k-1} \tag{56}$$

Now, if equation (54) is simplified further, it can be shown that

$$U_\Gamma^k = U_\Gamma^{k-1} + (S_1 + \gamma_1 \mathbf{I})^{-1}(G - SU_\Gamma^{k-1}) \tag{57}$$

Where $(S_1 + \gamma_1 \mathbf{I})$ is the preconditioner matrix P.

3 Coupling of heterogeneous problems

3.1 Problem 1

3.1.a Write down the equations in the wall assuming a plane stress behavior

Here the basic unknowns will be considered to be displacements u and v in the x and y axis directions. Strains, and hence stresses, can be expressed in terms of this displacements. The strains are written as

$$\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} \tag{58}$$

And, for the particular case of plane stress, the stresses are given by

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \varepsilon = \frac{E}{1-\nu^2} \begin{bmatrix} \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} \\ \frac{1-\nu}{2}(\frac{\partial u}{\partial y} + \frac{\partial x}{\partial x}) \end{bmatrix} \tag{59}$$

It remains to solve the equilibrium equation system, where X and Y are the external forces per unit volume

$$\begin{bmatrix} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + X \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y \end{bmatrix} = \mathbf{0} \tag{60}$$

The boundary conditions can be added as

$$\begin{bmatrix} u - \overline{u} \\ v - \overline{v} \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{0} \quad on \quad \Gamma_D \tag{61}$$

3.1.b Write down the equations for the beam modified because of the presence of the wall

The initial equation is

$$EI\frac{d^4v}{dx^4} = f \tag{62}$$

Now, the force that the wall transmits to the beam is $h\sigma_y(y = 0)$ (h being the thickness of the wall) and therefore the equation now will be

$$EI\frac{d^4v}{dx^4} = f - h\sigma_y(y = 0) = f - h\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right)_{y=0} \tag{63}$$

With the same boundary conditions plus the one that relates $v_w = v_b$ at y = 0.

3.1.c Obtain the adequate transmission conditions for v and the normal component of the traction on the wall at y = 0

Since the beam is supported by the wall, a clear transmission condition for v is $[\![v]\!] = 0$. As for the normal component of the traction, we have to compute the projection of the traction vector $\sigma \cdot \mathbf{n}$ on the y-axis direction, to have equilibrium on the y axis. That results in $[\![\mathbf{n} \cdot (\sigma(\mathbf{x}, \mathbf{0}) \cdot \mathbf{n})]\!]$.

3.1.d Suggest transmission conditions for u and the tangent component of the traction on the wall at y = 0. Discuss the implications if this component is not assumed to be zero.

There needs to be a transmission of the u component of the displacement such that $[\![u]\!] = 0$. The tangential component is found by projecting the traction vector $\sigma \cdot \mathbf{n}$ on the vector perpendicular to $\mathbf{n}$, so that now the transmission condition is $[\![\mathbf{t} \cdot (\sigma(\mathbf{x}, \mathbf{0}) \cdot \mathbf{n})]\!]$. If this is not assumed to be zero there we would not have equilibrium in the equations.

3.2 Problem 2

3.2.a Obtain the discrete version of the previous equation when space is discretized using finite elements. Relate the resulting matrices to those arising from the discretization of the Darcy and the Stokes problems separately.

When the partial differential equation is discretized by finite elements the discretization of the Steklov-Poincaré operator is the Schur complement obtained by eliminating all degrees of freedom inside the domain. Therefore, if denoting by $\mathbf{u_D^\Gamma}, \mathbf{u_S^\Gamma}$ the values of the unknowns of each problem on the interface, the matrix arising from the discretization of the operators will be

$$\begin{bmatrix} A_{DD} & A_{DS} \\ A_{SD} & A_{DS} \end{bmatrix} \begin{bmatrix} u_D^\Gamma \\ u_S^\Gamma \end{bmatrix} = \begin{bmatrix} f_D^\Gamma \\ f_S^\Gamma \end{bmatrix} \tag{64}$$

Now, if comparing the matrix with the one discretizing the other two problems separately, it is clear that we will get differences as those matrices consider the interior values. The Stokes problem has a very well-known matrix

$$\begin{bmatrix} K & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} u_S \\ p \end{bmatrix} = \begin{bmatrix} f_S \\ 0 \end{bmatrix} \tag{65}$$

As for the Darcy problem, the matrix system is not common to me so I will derive it through the weak form that is available in the notes

$$\begin{aligned} \int_\Omega \mathbf{w} \cdot (K^{-1}\mathbf{u_D} + \nabla\varphi) = \int_\Omega \mathbf{w} \cdot K^{-1}\mathbf{u_D} - \int_\Omega \varphi \boldsymbol{\nabla} \cdot \mathbf{w} + \int_{\partial\Omega} \varphi \mathbf{n} \cdot \mathbf{w} = \mathbf{0} \\ \int_\Omega q \boldsymbol{\nabla} \cdot u_D = \mathbf{0} \end{aligned} \tag{66}$$

Therefore the matrix system is very similar to that of the Stokes, although now the matrix that multiplies the velocity is computed differently,

$$\begin{bmatrix} B & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} u_D \\ \varphi \end{bmatrix} = \begin{bmatrix} f_D \\ 0 \end{bmatrix} \tag{67}$$

3.2.b Write down the matrix form of a Dirichlet-Neumann iteration-by-subdomain using the matrices of the Darcy and the Stokes problems

We solve first the Dirichlet problem on the Darcy sub-domain

$$\begin{aligned} \mathbf{B}\mathbf{u_D}^{k+1} + \mathbf{C}\varphi^{k+1} = \mathbf{f_D} - A_{D\Gamma}u_\Gamma^k \\ \mathbf{C^T}\mathbf{u_D}^{k+1} = 0 \end{aligned} \tag{68}$$

Now the Neumann ibs is (for the Stokes sub-domain)

$$\begin{bmatrix} \mathbf{A_{SS}} & \mathbf{B_{SS}} & \mathbf{A_{S\Gamma}} \\ \mathbf{B_{SS}}^T & 0 & \mathbf{B_{S\Gamma}} \\ \mathbf{A_{S\Gamma}} & \mathbf{B_{S\Gamma}}^T & \mathbf{A_{\Gamma\Gamma}^{(S)}} \end{bmatrix} \begin{bmatrix} \mathbf{U_S} \\ \mathbf{P_S} \\ \mathbf{U_\Gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{f_S} \\ \mathbf{0} \\ \mathbf{f_\Gamma}^S + \mathbf{f_\Gamma}^D - \mathbf{A_{\Gamma\Gamma}}^D\mathbf{u_\Gamma}^k - \mathbf{A_{\Gamma D}}^S\mathbf{u}^{k+1} \end{bmatrix} \tag{69}$$

3.2.c Identify the Richardson iteration for the algebraic problem in (a) resulting from (b)

4 Monolithic and partitioned schemes in time

4.1 Problem 1

4.1.a Discretize it using the finite element method (linear elements, element size h) for the discretization in space, and a BDF1 scheme for the discretization in time. Write down the weak form of the problem and the resulting matrix form of the problem, including the corresponding boundary integrals if necessary.

The discretization of the unknown u and its derivative are:

$$u(x,t) \approx \sum_j u_j(t) N_j(\mathbf{x}) \tag{70}$$

$$\partial_t u(x,t) \approx \sum_j \partial_t u_j(t) N_j(\mathbf{x}) \tag{71}$$

Given the boundary conditions of the problem and the fact that we have considered the test functions to be zero at the boundary, when the integration by parts is performed on the laplacian term, the integral over the boundary is zero, therefore the weak form of the problem is, after multiplying by the test function v is

$$(v, \partial_t u) + (\nabla v, \nabla u) = (v, f) \forall v \in V \tag{72}$$

And after discretizing this weak form using suitable spaces of functions $V_h \in V$ the resulting equation is

$$(v_h, \partial_t u_h) + (\nabla v_h, \nabla u_h) = (v_h, f) \forall v_h \in V_h \tag{73}$$

This results in the following matrix form

$$\mathbf{M} \partial_t \mathbf{u_h} + \mathbf{K} \mathbf{u_h} = \mathbf{f} \tag{74}$$

Where the matrices can be computed as

$$\mathbf{M}_{ij} = \int_0^1 N_i N_j dx \tag{75}$$

$$\mathbf{K}_{ij} = \int_0^1 \nabla N_i \nabla N_j dx \tag{76}$$

$$\mathbf{f}_{ij} = \int_0^1 N_i dx \tag{77}$$

Now, when using a first order Backward differences scheme to approximate the term $\partial_t \mathbf{u_h}$, it is reached an unconditionally stable result. First, let us write the implicit scheme, where the operators $\mathbf{K}, \mathbf{f}$ are evaluated at time $n+1$.

$$\mathbf{M}\partial_t \mathbf{U}^{n+1} + \mathbf{K}\mathbf{U}^{n+1} = \mathbf{f} \tag{78}$$

Now, with the BDF1 scheme,

$$\mathbf{M}\frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\delta t} + \mathbf{K}\mathbf{U}^{n+1} = \mathbf{f} \longrightarrow (\frac{\mathbf{M}}{\delta t} + \mathbf{K})\mathbf{U}^{n+1} = \mathbf{f} + \mathbf{M}\mathbf{U}^n/\delta t \tag{79}$$

And using the considered values

$$(\mathbf{M} + \mathbf{K})\mathbf{U}^{n+1} = \mathbf{1} + \mathbf{M}\mathbf{U}^n/\delta t \tag{80}$$

4.1.b Consider a domain decomposition approach for the previous problem. The left subdomain is composed of 2 elements (h = 0.2), while the right subdomain is composed of 3 elements (h = 0.2). Show that, if a monolithic approach is adopted, no boundary integrals are required at the interface. From now on, we denote the values at the nodes of the mesh as u0; u1, u2; u3, u4; u5. The interface is at u2.

The purpose in this exercise is that, using a monolithic approach, when constructing the weak form for each subdomain and summing both subdomains, the original equation is recovered due to the fact that the transmission conditions for this problem are equality of solution and fluxes at the interface, which in this case happens to be at $h = 0.4$. The weak form for both subdomains is, now considering that there is a contribution from the boundary:

$$(v_h^{(1)}, \partial_t u_h^{(1)}) + (\nabla v_h^{(1)}, \nabla u_h^{(1)}) - < v_h^{(1)}, \nabla u_h^{(1)} > = (v_h^{(1)}, f^{(1)}) \quad \forall v_h \in V_h \tag{81}$$

$$(v_h^{(2)}, \partial_t u_h^{(2)}) + (\nabla v_h^{(2)}, \nabla u_h^{(2)}) - < v_h^{(2)}, \nabla u_h^{(2)} > = (v_h^{(2)}, f^{(2)}) \quad \forall v_h \in V_h \tag{82}$$

Now, the transmission conditions that will allow us to recover the original equation are $< v_h^{(1)}, \nabla u_h^{(1)} > = - < v_h^{(2)}, \nabla u_h^{(2)} >$.

Therefore when summing up both equations,

$$(v_h^{(1)}, \partial_t u_h^{(1)}) + (v_h^{(2)}, \partial_t u_h^{(2)}) + (\nabla v_h^{(1)}, \nabla u_h^{(1)}) + (\nabla v_h^{(2)}, \nabla u_h^{(2)}) = (v_h^{(1)}, f^{(1)}) + (v_h^{(2)}, f^{(2)}) \quad \forall v_h \in V_h \tag{83}$$

Therefore showing that no boundary integrals are required at the interface.

4.1.c Obtain the algebraic form of the Dirichlet-to-Neumann operator for the left subdomain, departing from given values of u_i^n at time step n, and an interface value u_2^{n+1}.

The integral of the matrices multiplying the term U^{n+1} can be checked in any finite element technique book and yields for any element, considering linear shape functions and $\delta t = 1$,

$$\int_0^h \frac{dN_i}{dx}\frac{dN_j}{dx} + N_iN_j dx \longrightarrow \begin{bmatrix} 1/h + h/3 & -1/h + h/6 \\ -1/h + h/6 & 1/h + h/3 \end{bmatrix} \tag{84}$$

As for term f, we have

$$\int_0^h N_i dx \longrightarrow \begin{bmatrix} h/2 \\ h/2 \end{bmatrix} \tag{85}$$

The mapping of the surface temperature to the surface heat flux is a Poincaré–Steklov operator. This particular Poincare–Steklov operator is called the Dirichlet to Neumann (DtN) operator. Therefore the contribution of the heat flux on the boundary has to be added to f

$$\mathbf{f} = \begin{bmatrix} h/2 \\ h/2 + \phi^{n+1} \end{bmatrix} \tag{86}$$

Now the matrices are assembled takig into account that three nodes take part in the left subdomain, one of which is a Dirichlet node which can be removed its row and column, therefore, assembling the matrices,

$$\begin{bmatrix} 2(\frac{1}{h} + \frac{h}{3}) & \frac{h}{6} - \frac{1}{h} \\ \frac{h}{6} - \frac{1}{h} & \frac{1}{h} + \frac{h}{3} \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \end{bmatrix} = \begin{bmatrix} h \\ h/2 + \phi^{n+1} \end{bmatrix} + \begin{bmatrix} \frac{2h}{3} & \frac{h}{6} \\ \frac{h}{6} & \frac{2h}{3} \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \end{bmatrix} \tag{87}$$

Now as suggested we have to impose the term u_2^{n+1} as a known interface value (Dirichlet)

$$2(\frac{1}{h} + \frac{h}{3})U_1^{n+1} = h + \frac{2h}{3}U_1^n + \frac{h}{6}(U_2^n - U_2^{n+1}) - \frac{1}{h}U_2^{n+1} \tag{88}$$

And then, what remains to do is substituting this value into the second equation of the system of equations and isolate ϕ^{n+1} to obtain the Dirichlet to Neumann operator, where U_1^{n+1} is obtained from (88).

$$\phi^{n+1} = (\frac{h}{6} - \frac{1}{h})U_1^{n+1} + (\frac{1}{h} + \frac{h}{3})U_2^{n+1} - \frac{h}{2} + \frac{1}{h}(U_1^n - U_2^n) \tag{89}$$

4.1.d Obtain the algebraic form of the Neumann-to-Dirichlet operator for the right subdomain, departing from given values of u_i^n and an interface value for the fluxes $\phi^{n+1} = k\partial_x u^{n+1}$ at the coordinate of node 2.

The purpose now is to compute the interior values of the second sub-domain with the obtained value for the flux at the interface. The row and column of the u_5 node have also been removed. Now clearly, given the transmission condition, the flux will be accounting for a negative value in the equation. The matricial system is, after assembling the matrices obtained in the last part,

$$\begin{bmatrix} \frac{1}{h}+\frac{h}{3} & \frac{h}{6}-\frac{1}{h} & 0 \\ \frac{h}{6}-\frac{1}{h} & 2(\frac{1}{h}+\frac{h}{3}) & \frac{h}{6}-\frac{1}{h} \\ 0 & \frac{h}{6}-\frac{1}{h} & 2(\frac{1}{h}+\frac{h}{3}) \end{bmatrix} \begin{bmatrix} U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \end{bmatrix} = \begin{bmatrix} h/2-+\phi^{n+1} \\ h \\ h \end{bmatrix} + \frac{h}{3} \begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 2 & -1/2 \\ 0 & -1/2 & 2 \end{bmatrix} \begin{bmatrix} U_2^n \\ U_3^n \\ U_4^n \end{bmatrix} \tag{90}$$

With the value of the flux obtained in equation (89), the values at the interior at time step n+1 can be easily computed for the right sub-domain.

4.1.e Write down the iterative algorithm for a staggered approach applying Dirichlet boundary conditions at the interface to the left subdomain and Neumann boundary conditions at the interface for the right subdomain.

Let us write first the equations for the left and sub-domain,

$$\begin{bmatrix} 2(\frac{1}{h}+\frac{h}{3}) & \frac{h}{6}-\frac{1}{h} \\ \frac{h}{6}-\frac{1}{h} & \frac{1}{h}+\frac{h}{3} \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \end{bmatrix} = \begin{bmatrix} h \\ h/2+\phi^{n+1} \end{bmatrix} + \begin{bmatrix} \frac{2h}{3} & \frac{h}{6} \\ \frac{h}{6} & \frac{2h}{3} \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \end{bmatrix} \tag{91}$$

The first equation can be written as

$$\mathbf{A_x U_1}^{n+1} = \mathbf{F_x} + \mathbf{C_x U_1}^n - \mathbf{B_x}\widetilde{\mathbf{U_2}}^{n+1} \tag{92}$$

$$\begin{bmatrix} \frac{1}{h}+\frac{h}{3} & \frac{h}{6}-\frac{1}{h} & 0 \\ \frac{h}{6}-\frac{1}{h} & 2(\frac{1}{h}+\frac{h}{3}) & \frac{h}{6}-\frac{1}{h} \\ 0 & \frac{h}{6}-\frac{1}{h} & 2(\frac{1}{h}+\frac{h}{3}) \end{bmatrix} \begin{bmatrix} U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \end{bmatrix} = \begin{bmatrix} h/2-+\phi^{n+1} \\ h \\ h \end{bmatrix} + \frac{h}{3} \begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 2 & -1/2 \\ 0 & -1/2 & 2 \end{bmatrix} \begin{bmatrix} U_2^n \\ U_3^n \\ U_4^n \end{bmatrix} \tag{93}$$

The first equation of which can be rewritten in terms of the unknowns of the left sub-domain as

$$\mathbf{A_Y U_2}^{n+1} = \mathbf{F_Y} + \mathbf{C_Y U_2}^n - \mathbf{B_Y}\widetilde{\mathbf{U_1}}^{n+1} \tag{94}$$

Now the point of the staggered algorithm is to send the off-diagonal terms to the RHS through a prediction, therefore, by using a first order approximation, the system to be solved will be (first the left sub-domain and then the second sub-domain with the input of the first will be, by saying $\widetilde{\mathbf{U_1}}^{n+1} = \mathbf{U_1}^n, \widetilde{\mathbf{U_2}}^{n+1} = \mathbf{U_2}^n$:

$$\begin{cases} \mathbf{A_x U_1}^{n+1} = \mathbf{F_x} + \mathbf{C_x U_1}^n - \mathbf{B_x U_1}^n \\ \mathbf{A_Y U_2}^{n+1} = \mathbf{F_Y} + \mathbf{C_Y U_2}^n - \mathbf{B_Y U_2}^n \end{cases} \tag{95}$$

4.1.f Do the same for a substitution and an iteration by subdomains scheme.

Now the system is,

$$\begin{cases} \mathbf{A_x U_1}^{n+1} = \mathbf{F_x} + \mathbf{C_x U_1}^n - \mathbf{B_x}\widetilde{\mathbf{U_2}}^{n+1} \\ \mathbf{A_Y U_2}^{n+1} = \mathbf{F_Y} + \mathbf{C_Y U_2}^n - \mathbf{B_Y}\widetilde{\mathbf{U_1}}^{n+1} \end{cases} \tag{96}$$

In an iteration by subdomain, the algorithm would be,

(a) First, calculate the interior node of the left sub-domain, $\mathbf{U_1}^{n+1}$ at iteration number $i+1$ using values of the first iteration.

(b) Calculate the flux ϕ^{n+1} at iteration $i+1$ with the values computed.

(c) Solve $\mathbf{U_2}^{n+1}$ using the values that we have computed for $\mathbf{U_1}^{n+1}$.

Therefore now

$$\begin{cases} \mathbf{A_x U_1}^{n+1}|_{(i+1)} = \mathbf{F_x} + \mathbf{C_x U_1}^n - \mathbf{B_x}\widetilde{\mathbf{U_2}}^{n+1}|_{(i)} \\ \mathbf{A_Y U_2}^{n+1}|_{(i+1)} = \mathbf{F_Y} + \mathbf{C_Y U_2}^n - \mathbf{B_Y}\widetilde{\mathbf{U_1}}^{n+1}|_{(i)} \end{cases} \tag{97}$$

4.1.g Rewrite the algebraic system associated to the left subdomain (Dirichlet boundary conditions at the interface), using Nitsche's method for applying the boundary conditions. How does the condition number of the resulting system of equations vary with the penalty parameter α?

The initial problem in variational form is written as follows

$$(v, \partial_t u) + (\nabla v, \nabla u) = (v, f) \forall v \in V \tag{98}$$

The Nitche's method is an extension of the penalty method to make it symmetric and consistent. If we apply it to the left sub-domain we get the following variational form

$$(v_1, \partial_t u_1) + (\nabla v_1, \nabla u_1) + \alpha(v_1, u_1)_\Gamma - (v, n\cdot\nabla u_1)_\Gamma - (u_1, n\cdot\nabla u_1)_\Gamma = (v_1, f_1) + \alpha(v_1, u_D)_\Gamma - (u_D, n\cdot\nabla v_1)_\Gamma \quad \forall v \in V \tag{99}$$

This corresponds to the following modified system

$$(\mathbf{M}+\mathbf{K}+\mathbf{N})\mathbf{U}^{n+1} = \mathbf{f}+\mathbf{f_N}+\mathbf{M}\mathbf{U}^{n} \tag{100}$$

Where the N matrix has only contributions on the boundary term, therefore the system results

$$\begin{bmatrix} 2(\frac{1}{h}+\frac{h}{3}) & \frac{h}{6}-\frac{1}{h} \\ \frac{h}{6}-\frac{1}{h} & \frac{1}{h}+\frac{h}{3}+\frac{k}{h}(\alpha-1) \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \end{bmatrix} = \begin{bmatrix} h \\ h/2+\phi^{n+1}+\frac{k\overline{u_2}}{h}(\alpha-1) \end{bmatrix} + \begin{bmatrix} \frac{2h}{3} & \frac{h}{6} \\ \frac{h}{6} & \frac{2h}{3} \end{bmatrix} \begin{bmatrix} U_1^{n} \\ U_2^{n} \end{bmatrix} \tag{101}$$

The parameter α plays here an important role as it ensures stability of the model. As α increases, it is more stable. However, it has a counterpart which is that the condition number of the matrix increases with α, hence increasing the number of iterations needed.

5 Operator and splitting techniques

5.1 Problem 1

5.1.a Discretize it in space using finite elements (3 elements) and in time (finite differences, BDF1). Solve the first step of the problem, writing the solution as a function of the time step size δt.

The discretization of the unknown u and its derivative are:

$$u(x,t) \approx \sum_j u_j(t) N_j(\mathbf{x}) \tag{102}$$

$$\partial_t u(x,t) \approx \sum_j \partial_t u_j(t) N_j(\mathbf{x}) \tag{103}$$

Given the boundary conditions of the problem and the fact that we have considered the test functions to be zero at the boundary, when the integration by parts is performed on the laplacian term, the integral over the boundary is zero, therefore the weak form of the problem is, after multiplying by the test function v is

$$(v,\partial_t u) + (\nabla v, \nabla u) + (v, \nabla u) = (v,f) \forall v \in V \tag{104}$$

And after discretizing this weak form using suitable spaces of functions $V_h \in V$ the resulting equation is

$$(v_h,\partial_t u_h) + (\nabla v_h, \nabla u_h) + (v_h, \nabla u_h) = (v_h,f) \forall v_h \in V_h \tag{105}$$

This results in the following matrix form

$$\mathbf{M}\partial_t\mathbf{u_h} + \mathbf{K}\mathbf{u_h} + \mathbf{C}\mathbf{u_h} = \mathbf{F} \tag{106}$$

Where the matrices can be computed as

$$\mathbf{M}_{ij} = \int_0^1 N_i N_j dx \tag{107}$$

$$\mathbf{K}_{ij} = \int_0^1 \nabla N_i \nabla N_j dx \tag{108}$$

$$\mathbf{F}_{ij} = \int_0^1 N_i dx \tag{109}$$

$$\mathbf{C}_{ij} = \int_0^1 N_i \nabla N_j dx \tag{110}$$

Now, when using a first order Backward differences scheme to approximate the term $\partial_t\mathbf{u_h}$, it is reached an unconditionally stable result. First, let us write the implicit scheme, where the operators $\mathbf{K}, \mathbf{f}$ are evaluated at time $n+1$.

$$\mathbf{M}\partial_t\mathbf{U}^{n+1} + \mathbf{K}\mathbf{U}^{n+1} + \mathbf{C}\mathbf{U}^{n+1} = \mathbf{F}^{n+1} \tag{111}$$

Now, with the BDF1 scheme,

$$\mathbf{M}\frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\delta t} + \mathbf{K}\mathbf{U}^{n+1} + \mathbf{C}\mathbf{U}^{n+1} = \mathbf{F}^{n+1} \longrightarrow (\frac{\mathbf{M}}{\delta t} + \mathbf{K} + \mathbf{C})\mathbf{U}^{n+1} = \mathbf{F}^{n+1} + \frac{\mathbf{M}}{\delta t}\mathbf{U}^n \tag{112}$$

Let's find now the elementary matrices,

$$\mathbf{K_e} + \mathbf{M_e} = \int_0^h \frac{dN_i}{dx}\frac{dN_j}{dx} + N_i N_j dx \longrightarrow \begin{bmatrix} 1/h + h/3 & -1/h + h/6 \\ -1/h + h/6 & 1/h + h/3 \end{bmatrix} \tag{113}$$

$$\mathbf{C_e} = \int_0^h N_i \frac{dN_j}{dx} dx \longrightarrow \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \tag{114}$$

$$\mathbf{F_e} = \int_0^h N_i dx \longrightarrow \begin{bmatrix} h/2 \\ h/2 \end{bmatrix} \tag{115}$$

Now, when assembling the matrices, and given the boundary conditions, only the nodes in the middle will have to be solved. The reduced system will be, after assembling

$$(\begin{bmatrix} 2(1/h + \frac{h}{3\delta t}) & -1/h + \frac{h}{6\delta t} \\ -1/h + \frac{h}{6\delta t} & 2(1/h + \frac{h}{3\delta t}) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix})\mathbf{U}^{n+1} = \begin{bmatrix} h \\ h \end{bmatrix} + \begin{bmatrix} \frac{2h}{3} & \frac{h}{6} \\ \frac{h}{6} & \frac{2h}{3} \end{bmatrix}\mathbf{U}^n \tag{116}$$

Operating with the system, we achieve the following

$$\begin{bmatrix} 6+\frac{2}{9\delta t} & \frac{1}{18\delta t}-5/2 \\ \frac{1}{18\delta t}-7/2 & 6+\frac{2}{9\delta t} \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} + \begin{bmatrix} \frac{2}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{2}{9} \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \end{bmatrix} \tag{117}$$

The solution to this system, considering initial guess 0,

$$\begin{bmatrix} U_1^1 \\ U_2^1 \end{bmatrix} = \begin{bmatrix} \frac{6\delta t(51\delta t+1)}{2943\delta t^2+324\delta t+5} \\ \frac{6\delta t(57\delta t+1)}{2943\delta t^2+324\delta t+5} \end{bmatrix} \tag{118}$$

5.1.b Solve the same time step by using a first order operator splitting technique

The operator splitting technique divides the problem in two parts that solve a partial part of the problem, and a cycle is done until convergence reached. The two problems are the following,

$$\begin{aligned} (\frac{\mathbf{M}}{\delta t}+\mathbf{C})\mathbf{U_A}^{n+1} &= \frac{\mathbf{M}}{\delta t}\mathbf{U_A}^{n} \\ (\frac{\mathbf{M}}{\delta t}+\mathbf{K})\mathbf{U}^{n+1} &= \mathbf{F}^{n+1}+\frac{\mathbf{M}}{\delta t}\mathbf{U_A}^{n+1} \\ \mathbf{U_A}^{n+1} &= \mathbf{U}^{n+1} \longrightarrow repeat \end{aligned} \tag{119}$$

It is clear that $\mathbf{U_A}^{(1)}$ will be zero if we have considered that $\mathbf{U_A}^{(0)}$ is a vector of zeros. Then, to find the solution at the first time step, we only need to solve the following equation,

$$(\frac{\mathbf{M}}{\delta t}+\mathbf{K})\mathbf{U}^{n+1} = \mathbf{F}^{n+1} \tag{120}$$

Which, after computing wigh the values of the matrices that we have found, give the following easy solution compared to the monolitic solution

$$\begin{bmatrix} U_1^1 \\ U_2^1 \end{bmatrix} = \begin{bmatrix} \frac{6\delta t}{54\delta t+5} \\ \frac{6\delta t}{54\delta t+5} \end{bmatrix} \tag{121}$$

5.1.c Evaluate the error of the splitting approach with respect to the monolithic approach. Plot the splitting error vs. the time step size for $\delta t = 1$; $\delta t = 0.5$, $\delta t = 0.25$. Comment on the results

The solution of the monolithic is:

$$\delta t = 0.25 \longrightarrow \begin{bmatrix} U_1^1 \\ U_2^1 \end{bmatrix} = \begin{bmatrix} 0.0765 \\ 0.0874 \end{bmatrix}; \quad \delta t = 0.5 \longrightarrow \begin{bmatrix} U_1^1 \\ U_2^1 \end{bmatrix} = \begin{bmatrix} 0.0881 \\ 0.0980 \end{bmatrix}; \quad \delta t = 1 \longrightarrow \begin{bmatrix} U_1^1 \\ U_2^1 \end{bmatrix} = \begin{bmatrix} 0.0954 \\ 0.1064 \end{bmatrix} \tag{122}$$

The solution for the splitting technique is

$$\delta t = 0.25 \longrightarrow \begin{bmatrix} U_1^1 \\ U_2^1 \end{bmatrix} = \begin{bmatrix} 0.0811 \\ 0.0811 \end{bmatrix}; \quad \delta t = 0.5 \longrightarrow \begin{bmatrix} U_1^1 \\ U_2^1 \end{bmatrix} = \begin{bmatrix} 0.0938 \\ 0.0938 \end{bmatrix}; \quad \delta t = 1 \longrightarrow \begin{bmatrix} U_1^1 \\ U_2^1 \end{bmatrix} = \begin{bmatrix} 0.1017 \\ 0.1017 \end{bmatrix} \tag{123}$$

It is interesting to note that the operator splitting technique gives equal results for both nodes at the first time step for every time increment. This is due to the fact that if we consider an null initial guess for the solution, then the convective matrix (which non-symmetric and ultimately responsible for the different values at the nodes) has no effect on the solution. Then, in the following figure, it is possible to see the evolution of the error with different δt. It is concluded that the higher the time increment, the higher the error between methods, as could be expected.

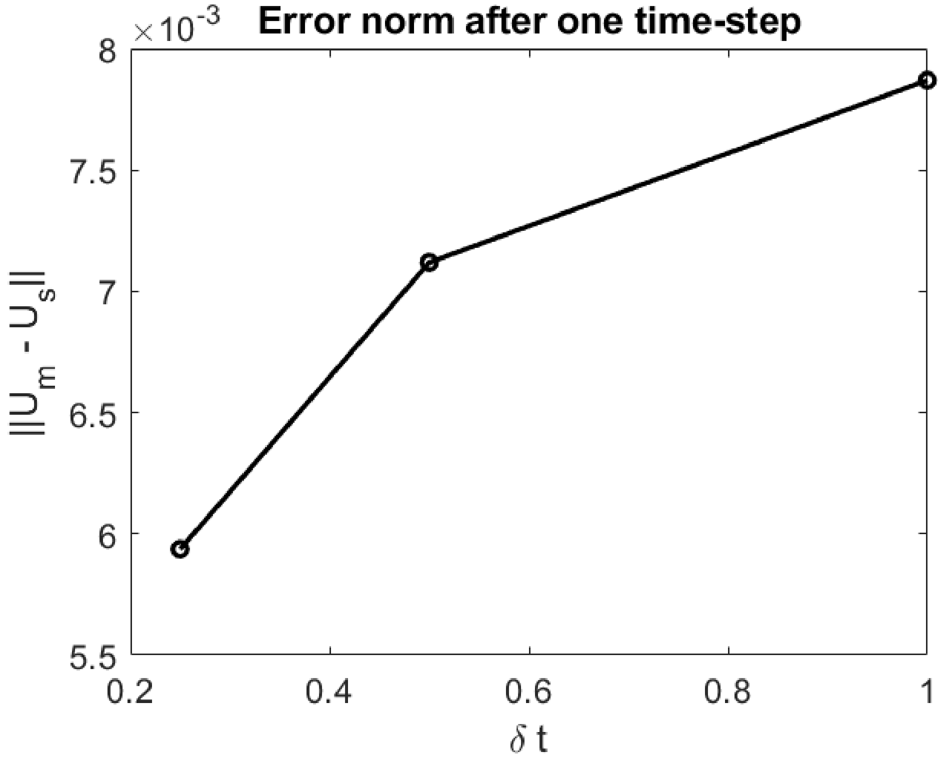

Figure 1: Caption

6 Fractional step methods

6.1 Problem 1

6.1.a Which is the optimal value for the α parameter?

In order to find which is the optimum value, we need to see which is the value of α that makes possible the recovering of the momentum equation.

Summing up the first and last equations, which are the extrapolated velocity equation and the correction equation for the velocity,

$$M\frac{1}{\delta t}(\hat{U}^{n+1}-U^{n}+U^{n+1}-\hat{U}^{n+1})+K(\hat{U}^{n+1}+\alpha U^{n+1}-\alpha\hat{U}^{n+1}) = f+G(P^{n+1}-\hat{P}^{n+1}+\hat{P}^{n+1}) \quad (124)$$

Now it is clear that when $\alpha = 1$ we recover the momentum equation, and this is the optimum value.

$$M\frac{1}{\delta t}(U^{n+1}-U^{n})+K(U^{n+1}) = f+G(P^{n+1}) \quad (125)$$

6.1.b What is the source of error of the scheme?

As a part of the fractional step methods, the Yosida scheme for solving the incompressible Navier-Stokes equations splits the original momentum equation into different steps, which involve intermediate velocity and pressure calculations as well as a correction for both magnitudes afterwards. Clearly, splitting the problem into parts will have an encompassed error which in this case affects the continuity equation [1]. Moreover, in Yoshida scheme a small perturbation is added to stabilize the solution when using a combination of interpolation degrees for the velocity and pressure elements, in case the LBB test is not passed. This way it is possible to have pressure and velocity interpolated at the same nodes. The higher this perturbation is, the more effect it will have on stability, but of course the perturbation diminishes the enforcement of the incompressibility condition. Hence the main source of the error for the Yosida scheme is the unsatisfied continuity equation [1].

7 ALE formulations

7.1 Problem 1

7.1.a Obtain the description of the property in terms of the ALE coordinates $(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$.

We start by introducing the mesh movement $\mathbf{x_m} = \mathbf{x_m}(\mathscr{X}, t)$. With this, For each initial position of the mesh nodes $\mathscr{X}$, it gives the position of the mesh nodes $\mathbf{x_m}$ at a given time instant. It traces the movement of the mesh. Let us now consider the description of a property in the ALE frame of reference $\gamma_{ALE}(\mathscr{X}(\mathbf{X},t),t)$. The description of the property will thus be,

$$\gamma(\mathscr{X},\mathscr{Y},\mathscr{Z},t) = [2(\mathscr{X}+\alpha t), \quad (\mathscr{Y}-\beta t)e^t, \quad \mathscr{Z}] \tag{126}$$

7.1.b Compute the velocity of the particles and the mesh velocity

The velocity of the particles is

$$\mathbf{v} := \frac{\partial \mathbf{x}(\mathbf{X},\mathbf{t})}{\partial t} = \begin{bmatrix} Xe^t \\ e^t \\ 0 \end{bmatrix} \tag{127}$$

As for the velocity of the mesh, it is the derivative of the equations of the mesh movement with respect to time,

$$\mathbf{v_m} = \frac{\partial \mathbf{x}(\mathscr{X},\mathbf{t})}{\partial t} = \begin{bmatrix} \alpha \\ -\beta \\ 0 \end{bmatrix} \tag{128}$$

7.1.c Compute the ALE description of the material temporal derivative of γ

Let us now consider the description of a property in the ALE frame of reference $\gamma_{ALE}(\mathscr{X}(\mathbf{X},t),t)$. Its material derivative is

$$\frac{d\gamma_{ALE}(\mathscr{X}(\mathbf{X},t),t)}{dt} = \frac{\partial \gamma_{ALE}(\mathscr{X},t)}{\partial t} + (\mathbf{v}-\mathbf{v_m})\cdot\nabla\gamma(\mathbf{x},t) \tag{129}$$

Let us first start with $\nabla\gamma(\mathbf{x},t)$

$$\nabla\gamma(\mathbf{x},t) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{130}$$

Then, the difference of the velocities needs to be computed with respect to the ALE coordinates

$$(\mathbf{v}-\mathbf{v_m}) = \begin{bmatrix} Xe^t+\alpha \\ e^t+\beta \\ 0 \end{bmatrix} = \begin{bmatrix} \mathscr{X}+\alpha(t-1) \\ e^t+\beta \\ 0 \end{bmatrix} \tag{131}$$

Eventually,

$$\frac{\partial \gamma_{ALE}(\mathscr{X},t)}{\partial t} = \begin{bmatrix} 2\alpha \\ -e^t(\beta t + \beta - \mathscr{Y}) \\ 0 \end{bmatrix} \tag{132}$$

$$\frac{d\gamma_{ALE}(\mathscr{X}(\mathbf{X},t),t)}{dt} = \begin{bmatrix} 2\alpha \\ -e^t(\beta t + \beta - \mathscr{Y}) \\ 0 \end{bmatrix} + \begin{bmatrix} \mathscr{X} + \alpha(t-1) \\ e^t + \beta \\ 0 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2(\mathscr{X} + \alpha t) \\ e^t(e^t - \beta t + \mathscr{Y}) \\ 0 \end{bmatrix} \tag{133}$$

7.2 Problem 2

7.2.a Write down the ALE form of the incompressible Navier-Stokes equations. Where (in time and space) is each of the terms of the equation evaluated? How are temporal derivatives computed?

In order to obtain the ALE formulation, let us first write the original strong form

$$\begin{aligned} \rho\frac{D\mathbf{u}}{Dt} - \boldsymbol{\nabla} \cdot \sigma(\mathbf{u}, p) = \mathbf{f} \\ \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \end{aligned} \tag{134}$$

Then, we have seen in the previous problem that the material derivative of a quantity in the ALE descriprion is obtained as

$$\frac{d\gamma_{ALE}(\mathscr{X}(\mathbf{X},t),t)}{dt} = \frac{\partial \gamma_{ALE}(\mathscr{X},t)}{\partial t} + (\mathbf{v} - \mathbf{v_m}) \cdot \nabla\gamma(\mathbf{x},t) \tag{135}$$

We can therefore obtain the material derivative of the velocity as

$$\frac{d\mathbf{u}_{ALE}(\mathscr{X}(\mathbf{X},t),t)}{dt} = \frac{\partial \mathbf{u}_{ALE}(\mathscr{X},t)}{\partial t} + (\mathbf{v} - \mathbf{v_m}) \cdot \nabla\mathbf{u}(\mathbf{x},t) \tag{136}$$

Eventually the equation is

$$\begin{aligned} \rho(\frac{\partial \mathbf{u}_{ALE}(\mathscr{X},t)}{\partial t} + (\mathbf{v} - \mathbf{v_m}) \cdot \nabla\mathbf{u}(\mathbf{x},t)) - \boldsymbol{\nabla} \cdot \sigma(\mathbf{u}, p) = \mathbf{f} \\ \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \end{aligned} \tag{137}$$

The temporal derivatives are computed at the moving nodes, as the difference of the values of the properties, but $\mathbf{u}_{ALE}$ is always evaluated at the same nodal coordinates even if the mesh is moving. As for the other terms that do not involve the temporal derivative, they are evaluated in the spatial coordinates in an Eulerian frame of reference.

7.3 Problem 3

7.3.a Do a bibliographical research on existing methods for the definition of the mesh movement in ALE formulations (Poisson problem, Elasticity problem, etc.). Describe the main advantages of each of these methods

In an Eulerian formulation, the mesh is fixed and the particles move through it, so mesh distortion can be avoided but the formulation may not be adequate to represent the movement of boundaries. With the aim of avoiding mesh distortion without needing to generate a new mesh, the arbitrary Lagrangian–Eulerian (ALE) kinematic description was used first to simulate fluid and fluid–structure interaction. The ALE formulation lead to an efficient treatment of fluid–structure interface, where nodes move independently of domain motion and excessive mesh distortion is avoided. Therefore the need for an adequate mesh-update strategy is underlined, and various automatic mesh-displacement prescription algorithms are reviewed. This includes mesh-regularization methods essentially based on geometrical concepts, as well as mesh-adaptation techniques aimed at optimizing the computational mesh according to some error indicator, with emphasis on particular issues related to the modeling of compressible and incompressible flow and nonlinear solid mechanics problems. This includes the treatment of convective terms in the conservation equations for mass, momentum, and energy [3]. Therefore, the methods are:

Mesh regularization, that keeps the regular mesh as regular as possible, avoiding mesh distortions and entanglement of the elements. This procedure decreases the numerical error due to mesh distortion. It requires that updated nodal coordinates are specified at each station of the calculation by using the mesh velocity. Then, the ALE formulation is able to reduce finite element distortion while representing the boundaries correctly. Since mesh and material movements are uncoupled in this formulation, convective terms appear in the balance of momentum equation, due to the relative motion between the material and the mesh, and equilibrium equations have twice as many unknowns as the number of equations [2].

Mesh adaption, whose purpose is to optimize the computational mesh to achieve better accuracy. This procedure decreases the numerical error due to mesh distortion. It requires that updated nodal coordinates are specified at each station of the calculation by using the mesh velocity. The algorithm includes an error indicator, and the mesh is modified to obtain an homogeneous distribution of the error over the computational domain.

Other techniques include **Transfinite mapping method**, which is a low computational cost procedure. **Laplacian Smoothing and Variational Methods**, presenting smooth distributions on the mesh nodes [4].

8 Fluid-Structure Interaction

8.1 Problem 1

8.1.a Describe the added mass effect problem for fluid structure interaction problems. When does it appear, what kind of problems suffer from it? What are the main methods for dealing with it?

The added mass effect is a phenomena occuring when the density of the fluid is similar to that of the solid. The operator acts as an additional mass on the degrees of freedom on the interface. It tends to occur on staggered problems that iterate from one equation to the other. Relaxtion methods, an in particular the Aitken scheme, which has been seen to provide good results in the numerical homework, are options to overcome this.This method, moreover, helps reaching the convergence faster.

8.2 Problem 2

8.2.a Consider the iteration by subdomain scheme for the heat transfer problem described in problem 1. Apply 2 iterations of the AITKEN relaxation scheme to it.

It has been seen previously that the problem has the matrix form:

$$(\mathbf{M}+\mathbf{K})\mathbf{U}^{n+1} = \mathbf{1} + \mathbf{M}\mathbf{U}^{n}/\delta t \tag{138}$$

The Neumann to Dirichlet method solves both sub-domains separately, and assigns the flux to the right boundary of the first sub-domain and the solution of the first sub-domain to the Dirichlet left value of the second. For that, at each iteration i we have for the first sub-domain:

$$(\mathbf{M_1}+\mathbf{K_1})\mathbf{U_1}(\mathbf{t^{n+1}})^k = \mathbf{f_1} + \mathbf{M_1}\mathbf{U_1}(\mathbf{t^n})/\delta t \tag{139}$$

Where the flux is calculated as follows

$$\frac{\partial U_1(t^{n+1})^k}{\partial x} = -\frac{\partial U_2(t^{n+1})^{k-1}}{\partial x} \tag{140}$$

As for the Dirichlet sub-domain, we have that

$$(\mathbf{M_2}+\mathbf{K_2})\mathbf{U_2}(\mathbf{t^{n+1}})^k = \mathbf{f_2} + \mathbf{M_2}\mathbf{U_2}(\mathbf{t^n})/\delta t \tag{141}$$

And now the Aitken algorithm comes as a way to calculate $\mathbf{U_2}(\mathbf{t^{n+1}})^k$ with information at two previous iterations. At time t^{n+1} the value at the interface is

$$u^i_{\Gamma 21} = wu^i_{\Gamma 12} + (1-w)u^{i-1}_{\Gamma 21} \tag{142}$$

And with the Aitken scheme,

$$w = u^i_{\Gamma 21} = \frac{u^{i-1}_{\Gamma 21} - u^{i-2}_{\Gamma 21}}{u^{i-1}_{\Gamma 21} - u^{i-2}_{\Gamma 21} + u^{i-1}_{\Gamma 12} - u^{i-2}_{\Gamma 12}} \tag{143}$$

Which implies that we need at least two iterations in order to start computing this value. Before that, w can simply be 0.5.

8.3 Problem 3

8.3.a Consider the monolithic (1 domain), transient (BDF1), finite element (linear elements, h = 1=4) approximation of the heat transfer equation in problem 1. Enforce the Dirichlet boundary conditions in x = 0 and x = 1 by using Lagrange multipliers. What is the form of the discrete system? What is the condition number of the resulting matrix?

The system of equations to be solved now will have the following form

$$\mathbf{M_e} + \mathbf{K_e} = \begin{bmatrix} \mathbf{A} & \mathbf{a^T} \\ \mathbf{a} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{U^{n+1}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f} + \mathbf{MU^n}/\delta t \\ \mathbf{b} \end{bmatrix} \tag{144}$$

With $\mathbf{A} = \mathbf{M}/\delta t + \mathbf{K}$

Where $\mathbf{a}$ is the vector of coefficients that assign the boundary value to the nodes (a vector of zeros and ones), and $\mathbf{b}$ contains the Dirichlet values. Since in this case is zero, we have

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \mathbf{a} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{145}$$

It has been seen that the sum of the elementary matrices is

$$\mathbf{M_e} + \mathbf{K_e} = \begin{bmatrix} 1/h + h/3 & -1/h + h/6 \\ -1/h + h/6 & 1/h + h/3 \end{bmatrix} \tag{146}$$

The good point in applying the Lagrange multipliers is that the boundary conditions are applied consistently. As for the condition number, if calculations are done and the matrices are assembled, considering $\Delta t = 1$, the condition matrix of the entire system is 38.3156 wheres that of matrix A is 73.0412. Therefore the first one will be easier to solve. However, not all matrix A has to be solved, only its reduced part, obtained after the first and last rows and columns are deleted. After doing that, the condition number is 5.3585, as is to be expected for a matrix with no zeros and only 3x3.

8.4 Problem 4

8.4.a Consider the monolithic (1 domain), transient (BDF1), finite element (linear elements, h = 0.25) approximation of the heat transfer equation in problem 1. Suppose that a level set function divides the domain into a high thermal conductivity (κ = 100) subdomain ($x \in$ [0; 0:4]) and a low thermal conductivity (κ = 1) subdomain ($x \in$ (0:4; 1]). Build the system matrix for this problem. Take into account the need for subintegrating the element cut by the level set function.

For this problem, there are four elements, and the one which contains the points $x = 0.4$ has two different values for k. For this reason, it will have to be integrated in different regions. We have for the different element

$$\mathbf{K_1} = \int_0^h 100\frac{dN_i}{dx}\frac{dN_j}{dx} = 100\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \tag{147}$$

$$\mathbf{K_2} = 100\int_h^{0.4} \frac{dN_i}{dx}\frac{dN_j}{dx} + \int_{0.4}^{2h} \frac{dN_i}{dx}\frac{dN_j}{dx} = 400\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}(0.15) + 4\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}(0.1) = 241.6\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(0.1) \tag{148}$$

$$\mathbf{K_3} = \int_0^h \frac{dN_i}{dx}\frac{dN_j}{dx} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \tag{149}$$

$$\mathbf{K_4} = \int_0^h \frac{dN_i}{dx}\frac{dN_j}{dx} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \tag{150}$$

As for the mass matrix of each element, it has been calculated before and is

$$\mathbf{M^e} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}/12 \tag{151}$$

Then there is only to sum the contribution of these two matrix to generate the global matrix, with the same source term as before.

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