Coupled Problems Assignment

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Given the weak form of the problem in 1D with homogeneous Dirchlet conditions (clamped beam at both ends):

$$EI \int_0^L \frac{d^2\delta v}{dx^2} \frac{d^2 v}{dx^2} = \int_0^L \delta v f$$

for all δv such that $\delta v(0) = \delta v(L) = 0$, $\frac{d\delta v}{dx}(0) = \frac{d\delta v}{dx}(L) = 0$

(a) The space of functions for v and δv are:

$$\boxed{v \in \mathcal{H}^2(\Omega) \quad \text{such that} \quad v|_{\partial\Omega} = \left.\frac{dv}{dx}\right|_{\partial\Omega} = 0}$$

$$\boxed{\delta v \in \mathcal{H}^2(\Omega) \quad \text{such that} \quad \delta v|_{\partial\Omega} = \left.\frac{d\delta v}{dx}\right|_{\partial\Omega} = 0}$$

where $\mathcal{H}^2(\Omega)$ is the space of functions that are defined in Ω such that a function plus its first-order and second-order derivatives are square integrable, namely:

$$\mathcal{H}^2(\Omega) := \left\{ v : \Omega \to \mathbb{R} \ \middle| \ \int_\Omega v^2 < \infty, \ \int_\Omega |\nabla v|^2 < \infty, \ \int_\Omega |\Delta v|^2 < \infty \right\}$$

The reason is that for the integral, appearing on the left-hand-side of the weak form, to be bounded, the second-order derivatives of v and δv have to be square integrable.

(b) Based on the regularity requirements, v and $\frac{dv}{dx}$ have to be continuous. Therefore, the transmission conditions at point P implied by the regularity requirements are:

$$\boxed{[\![v]\!]_P = 0} \longrightarrow \text{1st transmission condition - strong continuity}$$

$$\boxed{[\![\frac{dv}{dx}]\!]_P = 0} \longrightarrow \text{2nd transmission condition - strong continuity}$$

where,

$$[\![v]\!]_P = \lim_{\varepsilon \to 0} \left[v(P + \varepsilon) - v(P - \varepsilon) \right]$$

$$[\![\frac{dv}{dx}]\!]_P = \lim_{\varepsilon \to 0} \left[\frac{dv}{dx}(P + \varepsilon) - \frac{dv}{dx}(P - \varepsilon) \right]$$

Justification in 1D:
if v and $\frac{dv}{dx}$ are discontinuous at P, therefore, the second-order derivative can be written

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in terms of the variable v, near the discontinuity, as:

$$\frac{d^2v^\varepsilon}{dx^2} = \begin{cases} \dfrac{d^2v}{dx^2} & x < P - \varepsilon \\[2ex] \dfrac{1}{2\varepsilon}\Big(v(P+\varepsilon) - 2v(P) + v(P-\varepsilon)\Big) & P - \varepsilon < x < P + \varepsilon \\[2ex] \dfrac{d^2v}{dx^2} & x < P + \varepsilon \end{cases} \qquad (1)$$

or alternatively in terms of the first-order derivative, near the discontinuity, as:

$$\frac{d^2v^\varepsilon}{dx^2} = \begin{cases} \dfrac{d^2v}{dx^2} & x < P - \varepsilon \\[2ex] \dfrac{1}{2\varepsilon}\Big(\dfrac{dv}{dx}(P+\varepsilon) - \dfrac{dv}{dx}(P-\varepsilon)\Big) & P - \varepsilon < x < P + \varepsilon \\[2ex] \dfrac{d^2v}{dx^2} & x < P + \varepsilon \end{cases} \qquad (2)$$

Using the approximation given by (1), the square integral of the second-order derivative of v is then evaluated as:

$$\int_0^L \left(\frac{d^2v^\varepsilon}{dx^2}\right)^2 = \int_0^{P-\varepsilon} \left(\frac{d^2v}{dx^2}\right)^2 + \int_{P-\varepsilon}^{P+\varepsilon} \left(\frac{1}{2\varepsilon}\Big(v(P+\varepsilon) - 2v(P) + v(P-\varepsilon)\Big)\right)^2 + \int_{P+\varepsilon}^L \left(\frac{d^2v}{dx^2}\right)^2$$

$$= \int_0^{P-\varepsilon} \left(\frac{d^2v}{dx^2}\right)^2 + \frac{1}{2\varepsilon}\Big(v(P+\varepsilon) - 2v(P) + v(P-\varepsilon)\Big)^2 + \int_{P+\varepsilon}^L \left(\frac{d^2v}{dx^2}\right)^2$$

$$\varepsilon \to 0 \quad = \infty$$

This proves that if the variable v is discontinuous, then the second-order derivative $\dfrac{d^2v}{dx^2}$ is not square integrable, i.e. $v \notin \mathcal{H}^2$.

Using the approximation given by (2), the square integral of the second-order derivative of v is then evaluated as:

$$\int_0^L \left(\frac{d^2v^\varepsilon}{dx^2}\right)^2 = \int_0^{P-\varepsilon} \left(\frac{d^2v}{dx^2}\right)^2 + \int_{P-\varepsilon}^{P+\varepsilon} \left(\frac{1}{2\varepsilon}\Big(\frac{dv}{dx}(P+\varepsilon) - \frac{dv}{dx}(P-\varepsilon)\Big)\right)^2 + \int_{P+\varepsilon}^L \left(\frac{d^2v}{dx^2}\right)^2$$

$$= \int_0^{P-\varepsilon} \left(\frac{d^2v}{dx^2}\right)^2 + \frac{1}{2\varepsilon}\Big(\frac{dv}{dx}(P+\varepsilon) - \frac{dv}{dx}(P-\varepsilon)\Big)^2 + \int_{P+\varepsilon}^L \left(\frac{d^2v}{dx^2}\right)^2$$

$$\varepsilon \to 0 \quad = \infty$$

This proves that if the first-order derivative $\dfrac{dv}{dx}$ is discontinuous, then the second-order derivative $\dfrac{d^2v}{dx^2}$ is not square integrable, i.e. $v \notin \mathcal{H}^2$.

(c) Considering the strong form of the equation:

$$EI\frac{d^4v}{dx^4} = f$$

Multiplying by a test function δv, and integrating over a domain Ω yields the following weak form:

$$\int_\Omega \delta v\, EI\frac{d^4v}{dx^4} = \int_\Omega \delta v f$$

Integrating by parts yields:

$$-\int_\Omega \frac{d\delta v}{dx}EI\frac{d^3v}{dx^3} + \int_\Omega \frac{d}{dx}\left(\delta v\, EI\frac{d^3v}{dx^3}\right) = \int_\Omega \delta v f$$

Applying divergence theorem, the second term on the LHS will be a boundary integral:

$$-\int_\Omega \frac{d\delta v}{dx}EI\frac{d^3v}{dx^3} + \int_{\partial\Omega} \delta v\, EI\frac{d^3v}{dx^3}n = \int_\Omega \delta v f \tag{3}$$

For a domain $\Omega = \Omega_1 \cup \Omega_2$ with an interface $\Gamma = \Omega_1 \cap \Omega_2$, the equation is written for each domain as follows:

Domain Ω_1:

$$-\int_{\Omega_1} \frac{d\delta v}{dx}EI\frac{d^3v}{dx^3} + \int_{\partial\Omega_1} \delta v\, EI\frac{d^3v}{dx^3}n_1 = \int_{\Omega_1} \delta v f$$

splitting the second term on LHS,

$$-\int_{\Omega_1} \frac{d\delta v}{dx}EI\frac{d^3v}{dx^3} + \int_{\partial\Omega_1\cap\partial\Omega} \delta v\, EI\frac{d^3v}{dx^3}n_1 + \int_{\partial\Omega_1\cap\Gamma} \delta v\, EI\frac{d^3v}{dx^3}n_1 = \int_{\Omega_1} \delta v f \tag{4}$$

Domain Ω_2:

$$-\int_{\Omega_2} \frac{d\delta v}{dx}EI\frac{d^3v}{dx^3} + \int_{\partial\Omega_2} \delta v\, EI\frac{d^3v}{dx^3}n_2 = \int_{\Omega_2} \delta v f$$

splitting the second term on LHS,

$$-\int_{\Omega_2} \frac{d\delta v}{dx}EI\frac{d^3v}{dx^3} + \int_{\partial\Omega_2\cap\partial\Omega} \delta v\, EI\frac{d^3v}{dx^3}n_2 + \int_{\partial\Omega_2\cap\Gamma} \delta v\, EI\frac{d^3v}{dx^3}n_2 = \int_{\Omega_2} \delta v f \tag{5}$$

Summing equations (4) and (5) should return (3), this means that the extra terms are equal to zero, therefore:

$$\int_{\partial\Omega_1\cap\Gamma} \delta v\, EI\frac{d^3v}{dx^3}n_1 + \int_{\partial\Omega_2\cap\Gamma} \delta v\, EI\frac{d^3v}{dx^3}n_2 = 0$$

which is simply written as:

$$\int_\Gamma \delta v\left((EI)_1\frac{d^3v}{dx^3}n_1 + (EI)_2\frac{d^3v}{dx^3}n_2\right) = 0 \quad \Longrightarrow \quad \left[\!\left[EI\frac{d^3v}{dx^3}n\right]\!\right]_\Gamma = 0$$

Particularizing for the problem at hand, where the $\Gamma = P$, $\Omega_1 = [0, P]$, $\Omega_2 = (P, L]$ and $n_1 = -n_2 = 1$, yields the third transmission condition:

$$\boxed{[\![EI\frac{d^3v}{dx^3}]\!]_P = 0} \longrightarrow \text{3rd transmission condition - weak continuity}$$

Integrating (3) by parts again yields:

$$\int_\Omega \frac{d^2\delta v}{dx^2}EI\frac{d^2v}{dx^2} - \int_\Omega \frac{d}{dx}\left(\frac{d\delta v}{dx}EI\frac{d^2v}{dx^2}\right) + \int_{\partial\Omega} \delta v\ EI\frac{d^3v}{dx^3}n = \int_\Omega \delta vf$$

Applying divergence theorem, the second term on the LHS will be a boundary integral:

$$\int_\Omega \frac{d^2\delta v}{dx^2}EI\frac{d^2v}{dx^2} - \int_{\partial\Omega} \frac{d\delta v}{dx}EI\frac{d^2v}{dx^2}n + \int_{\partial\Omega} \delta v\ EI\frac{d^3v}{dx^3}n = \int_\Omega \delta vf \tag{6}$$

For a domain $\Omega = \Omega_1 \cup \Omega_2$ with an interface $\Gamma = \Omega_1 \cap \Omega_2$, the equation is written for each domain as follows:

Domain Ω_1:

$$\int_{\Omega_1} \frac{d^2\delta v}{dx^2}EI\frac{d^2v}{dx^2} - \int_{\partial\Omega_1} \frac{d\delta v}{dx}EI\frac{d^2v}{dx^2}n_1 + \int_{\partial\Omega_1} \delta v\ EI\frac{d^3v}{dx^3}n_1 = \int_{\Omega_1} \delta vf$$

splitting the second and third terms on LHS,

$$\int_{\Omega_1} \frac{d^2\delta v}{dx^2}EI\frac{d^2v}{dx^2} - \int_{\partial\Omega_1\cap\partial\Omega} \frac{d\delta v}{dx}EI\frac{d^2v}{dx^2}n_1 - \int_{\partial\Omega_1\cap\Gamma} \frac{d\delta v}{dx}EI\frac{d^2v}{dx^2}n_1$$
$$+ \int_{\partial\Omega_1\cap\partial\Omega} \delta v\ EI\frac{d^3v}{dx^3}n_1 + \int_{\partial\Omega_1\cap\Gamma} \delta v\ EI\frac{d^3v}{dx^3}n_1 = \int_{\Omega_1} \delta vf \tag{7}$$

Domain Ω_2:

$$\int_{\Omega_2} \frac{d^2\delta v}{dx^2}EI\frac{d^2v}{dx^2} - \int_{\partial\Omega_2} \frac{d\delta v}{dx}EI\frac{d^2v}{dx^2}n_2 + \int_{\partial\Omega_2} \delta v\ EI\frac{d^3v}{dx^3}n_2 = \int_{\Omega_2} \delta vf$$

splitting the second and third terms on LHS,

$$\int_{\Omega_2} \frac{d^2\delta v}{dx^2}EI\frac{d^2v}{dx^2} - \int_{\partial\Omega_2\cap\partial\Omega} \frac{d\delta v}{dx}EI\frac{d^2v}{dx^2}n_2 - \int_{\partial\Omega_2\cap\Gamma} \frac{d\delta v}{dx}EI\frac{d^2v}{dx^2}n_2$$
$$+ \int_{\partial\Omega_2\cap\partial\Omega} \delta v\ EI\frac{d^3v}{dx^3}n_2 + \int_{\partial\Omega_2\cap\Gamma} \delta v\ EI\frac{d^3v}{dx^3}n_2 = \int_{\Omega_2} \delta vf \tag{8}$$

Summing equations (7) and (8) should return (6), this means that the extra terms are equal to zero, therefore:

$$-\int_{\partial\Omega_1\cap\Gamma} \frac{d\delta v}{dx}EI\frac{d^2v}{dx^2}n_1 - \int_{\partial\Omega_2\cap\Gamma} \frac{d\delta v}{dx}EI\frac{d^2v}{dx^2}n_2 + \int_{\partial\Omega_1\cap\Gamma} \delta v\ EI\frac{d^3v}{dx^3}n_1 + \int_{\partial\Omega_2\cap\Gamma} \delta v\ EI\frac{d^3v}{dx^3}n_2 = 0$$

which is simply written as:

$$-\int_{\Gamma} \frac{d\delta v}{dx}\Big((EI)_1\frac{d^2v}{dx^2}n_1 + (EI)_2\frac{d^2v}{dx^2}n_2\Big) + \int_{\Gamma} \delta v\Big((EI)_1\frac{d^3v}{dx^3}n_1 + (EI)_2\frac{d^3v}{dx^3}n_2\Big) = 0$$

Noting that the second term is zero (third transmission condition), therefore:

$$\int_{\Gamma} \frac{d\delta v}{dx}\Big((EI)_1\frac{d^2v}{dx^2}n_1 + (EI)_2\frac{d^2v}{dx^2}n_2\Big) = 0 \quad \Longrightarrow \quad [\![EI\frac{d^2v}{dx^2}n]\!]_{\Gamma} = 0$$

Particularizing for the problem at hand, where the $\Gamma = P$, $\Omega_1 = [0, P]$, $\Omega_2 = (P, L]$ and $n_1 = -n_2 = 1$, yields the fourth transmission condition:

$$\boxed{[\![EI\frac{d^2v}{dx^2}]\!]_P = 0} \longrightarrow \text{4th transmission condition - weak continuity}$$

1.2 Question 2: Maxwell problem

The Maxwell problem consists in finding a vector field $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^3$ such that

$$\nu \, \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega$$
$$\boldsymbol{n} \times \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega$$

where $\nu > 0$, $\boldsymbol{f}$ is a divergence free force field and $\boldsymbol{n}$ the unit external normal. Equation $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ is in fact redundant.

(a) To obtain the variational or weak form, pre-multiply by a vector test function $\delta\boldsymbol{u}$ and integrate over the domain Ω:

$$\int_\Omega \delta\boldsymbol{u} \cdot (\nu \, \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u}) = \int_\Omega \delta\boldsymbol{u} \cdot \boldsymbol{f} \tag{9}$$

For the integration by parts, first recall that:

$$\boldsymbol{\nabla}\cdot(\delta\boldsymbol{u} \times \boldsymbol{\nabla} \times \boldsymbol{u}) = \boldsymbol{\nabla}\cdot(\delta\boldsymbol{u} \times \boldsymbol{w}) = \boldsymbol{\nabla}\cdot\boldsymbol{v} = \frac{\partial v_i}{\partial x_i}$$

where,

$$v_i = (\delta\boldsymbol{u} \times \boldsymbol{w})_i = \varepsilon_{ijk} \, \delta u_j \, w_k,$$
$$w_k = (\boldsymbol{\nabla} \times \boldsymbol{u})_k = \varepsilon_{klm} \, \frac{\partial u_m}{\partial x_l}$$

therefore,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\delta\boldsymbol{u} \times \boldsymbol{\nabla} \times \boldsymbol{u}) &= \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk} \, \delta u_j \, \varepsilon_{klm} \, \frac{\partial u_m}{\partial x_l}\right) \\
&= \varepsilon_{ijk} \, \frac{\partial \delta u_j}{\partial x_i} \, \varepsilon_{klm} \, \frac{\partial u_m}{\partial x_l} + \varepsilon_{ijk} \, \delta u_j \, \frac{\partial}{\partial x_i}\left(\varepsilon_{klm} \, \frac{\partial u_m}{\partial x_l}\right) \\
&= \varepsilon_{kij} \, \frac{\partial \delta u_j}{\partial x_i} \, \varepsilon_{klm} \, \frac{\partial u_m}{\partial x_l} - \delta u_j \, \varepsilon_{jik} \, \frac{\partial}{\partial x_i}\left(\varepsilon_{klm} \, \frac{\partial u_m}{\partial x_l}\right) \\
&= \left(\varepsilon_{kij} \, \frac{\partial \delta u_j}{\partial x_i}\right)\left(\varepsilon_{klm} \, \frac{\partial u_m}{\partial x_l}\right) - \delta u_j \, \varepsilon_{jik} \, \frac{\partial}{\partial x_i}\left(\varepsilon_{klm} \, \frac{\partial u_m}{\partial x_l}\right) \\
&= (\boldsymbol{\nabla} \times \delta\boldsymbol{u})_k \, (\boldsymbol{\nabla} \times \boldsymbol{u})_k - \delta u_j \left(\varepsilon_{jik} \, \frac{\partial}{\partial x_i}(\boldsymbol{\nabla} \times \boldsymbol{u})_k\right) \\
&= (\boldsymbol{\nabla} \times \delta\boldsymbol{u})_k \, (\boldsymbol{\nabla} \times \boldsymbol{u})_k - \delta u_j \, (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u})_j \\
&= (\boldsymbol{\nabla} \times \delta\boldsymbol{u}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) - \delta\boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u})
\end{aligned}$$

which is re-written as:

$$\delta\boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u}) = (\boldsymbol{\nabla} \times \delta\boldsymbol{u}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) - \boldsymbol{\nabla}\cdot(\delta\boldsymbol{u} \times \boldsymbol{\nabla} \times \boldsymbol{u}) \tag{10}$$

Using the information given by (10), integral (9) is written as:

$$\int_\Omega (\boldsymbol{\nabla} \times \delta\boldsymbol{u}) \cdot (\nu \, \boldsymbol{\nabla} \times \boldsymbol{u}) - \int_\Omega \boldsymbol{\nabla}\cdot(\nu \, \delta\boldsymbol{u} \times \boldsymbol{\nabla} \times \boldsymbol{u}) = \int_\Omega \delta\boldsymbol{u} \cdot \boldsymbol{f}$$

Applying divergence theorem to the second term on the LHS yields:

$$\int_\Omega (\boldsymbol{\nabla} \times \delta\boldsymbol{u}) \cdot (\nu\, \boldsymbol{\nabla} \times \boldsymbol{u}) - \int_{\partial\Omega} \boldsymbol{n} \cdot (\nu\, \delta\boldsymbol{u} \times \boldsymbol{\nabla} \times \boldsymbol{u}) = \int_\Omega \delta\boldsymbol{u} \cdot \boldsymbol{f} \qquad (11)$$

Recalling the following relation for vectors:

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u})$$

therefore, equation (11) is written as:

$$\int_\Omega (\boldsymbol{\nabla} \times \delta\boldsymbol{u}) \cdot (\nu\, \boldsymbol{\nabla} \times \boldsymbol{u}) - \int_{\partial\Omega} (\nu\, \boldsymbol{\nabla} \times \boldsymbol{u}) \cdot (\boldsymbol{n} \times \delta\boldsymbol{u}) = \int_\Omega \delta\boldsymbol{u} \cdot \boldsymbol{f} \qquad (12)$$

Since $\boldsymbol{n} \times \boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$ (Dirchlet boundary only on $\partial\Omega$, i.e. $\boldsymbol{n} \times \delta\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$), therefore the weak form of the problem is written as:

$$\boxed{\int_\Omega (\boldsymbol{\nabla} \times \delta\boldsymbol{u}) \cdot (\nu\, \boldsymbol{\nabla} \times \boldsymbol{u}) = \int_\Omega \delta\boldsymbol{u} \cdot \boldsymbol{f}} \qquad (13)$$

where the space of functions for $\boldsymbol{u}$ and $\delta\boldsymbol{u}$ is:

$$\boxed{\boldsymbol{u} \in \mathcal{H}^{curl}(\Omega) \quad \text{such that} \quad (\boldsymbol{n} \times \boldsymbol{u})|_{\partial\Omega} = \boldsymbol{0}}$$

$$\boxed{\delta\boldsymbol{u} \in \mathcal{H}^{curl}(\Omega) \quad \text{such that} \quad (\boldsymbol{n} \times \delta\boldsymbol{u})|_{\partial\Omega} = \boldsymbol{0}}$$

where $\mathcal{H}^{curl}(\Omega)$ is the space of vector functions of dimension (d) that are defined in Ω such that a function plus its curl are square integrable (belong to $[\mathcal{L}_2(\Omega)]^d$), namely:

$$\mathcal{H}^{curl}(\Omega) := \left\{ \boldsymbol{u} : \Omega \to \mathbb{R}^d \ \middle|\ \boldsymbol{u} \in \left[\mathcal{L}_2(\Omega)\right]^d,\ \boldsymbol{\nabla} \times \boldsymbol{u} \in \left[\mathcal{L}_2(\Omega)\right]^d \right\}$$

The reason is that for the integrals, appearing in the weak form (13), to be bounded, $\boldsymbol{\nabla} \times \boldsymbol{u}$, $\boldsymbol{\nabla} \times \delta\boldsymbol{u}$ and $\delta\boldsymbol{u}$ have to be square integrable.

(b) The transmission conditions that satisfy the regularity requirement is obtained by considering that:

$$\int_\Omega |\boldsymbol{\nabla} \times \boldsymbol{u}|^2 \leq \infty$$

$$\boxed{[\![\boldsymbol{n} \times \boldsymbol{u}]\!]_\Gamma = \boldsymbol{0}} \longrightarrow \text{1st transmission condition - strong continuity}$$

Considering the simple schematic in Figure 1, the jump operator is:

$$[\![\boldsymbol{n} \times \boldsymbol{u}]\!]_\Gamma = \lim_{\varepsilon \to \boldsymbol{0}} \left[\boldsymbol{n} \times \boldsymbol{u}(\boldsymbol{x}_o + \boldsymbol{\varepsilon}) - \boldsymbol{n} \times \boldsymbol{u}(\boldsymbol{x}_o - \boldsymbol{\varepsilon}) \right]$$

Justification:
If $\boldsymbol{n} \times \boldsymbol{u}$ is discontinuous at Γ, therefore, we can define $\boldsymbol{\nabla} \times \boldsymbol{u}$ as:

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \begin{cases} \boldsymbol{\nabla} \times \boldsymbol{u} & \boldsymbol{x} \in \Omega_1 \\[2mm] \dfrac{1}{2\varepsilon}\left(\boldsymbol{n} \times \boldsymbol{u}(\boldsymbol{x}_o + \boldsymbol{\varepsilon}) - \boldsymbol{n} \times \boldsymbol{u}(\boldsymbol{x}_o - \boldsymbol{\varepsilon}) \right) & \boldsymbol{x}_o + \boldsymbol{\varepsilon} < \boldsymbol{x} < \boldsymbol{x}_o - \boldsymbol{\varepsilon} \\[2mm] \boldsymbol{\nabla} \times \boldsymbol{u} & \boldsymbol{x} \in \Omega_2 \end{cases} \qquad (14)$$
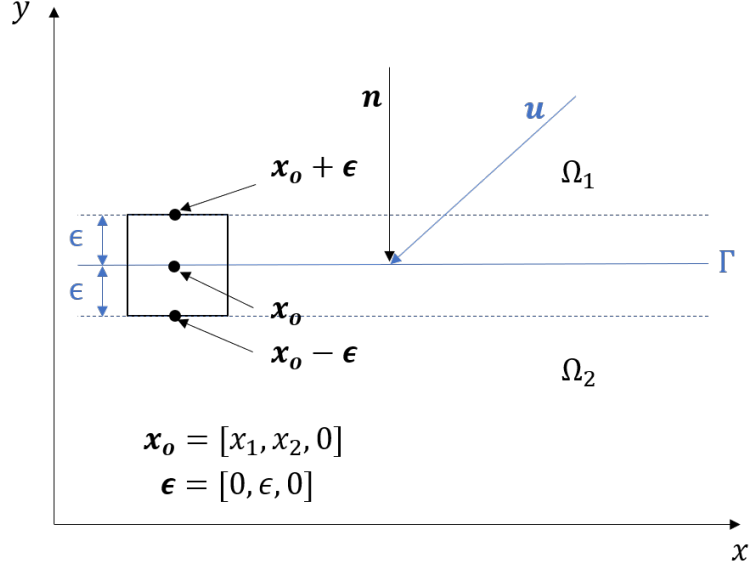
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Figure 1: Maxwell problem interface schematic

Using the approximation given by (14), the square integral of $\boldsymbol{\nabla} \times \boldsymbol{u}$ is then evaluated as:

$$\int_{\Omega} \left(\boldsymbol{\nabla} \times \boldsymbol{u}\right)^2 = \int_{\Omega_1} \left(\boldsymbol{\nabla} \times \boldsymbol{u}\right)^2 + \int_{\boldsymbol{x}_o+\varepsilon}^{\boldsymbol{x}_o-\varepsilon} \left(\frac{1}{2\varepsilon}\left[\boldsymbol{n} \times \boldsymbol{u}(\boldsymbol{x}_o + \varepsilon) - \boldsymbol{n} \times \boldsymbol{u}(\boldsymbol{x}_o - \varepsilon)\right]\right)^2 + \int_{\Omega_2} \left(\boldsymbol{\nabla} \times \boldsymbol{u}\right)^2$$

$$= \int_{\Omega_1} \left(\boldsymbol{\nabla} \times \boldsymbol{u}\right)^2 + \frac{1}{2\varepsilon}\left[\boldsymbol{n} \times \boldsymbol{u}(\boldsymbol{x}_o + \varepsilon) - \boldsymbol{n} \times \boldsymbol{u}(\boldsymbol{x}_o - \varepsilon)\right]^2 + \int_{\Omega_2} \left(\boldsymbol{\nabla} \times \boldsymbol{u}\right)^2$$

$$\varepsilon \to 0 \quad = \infty$$

This proves that if $\boldsymbol{n} \times \boldsymbol{u}$ is discontinuous, then $\boldsymbol{\nabla} \times \boldsymbol{u}$ is not square integrable, i.e. $\boldsymbol{u} \notin \mathcal{H}^{curl}$.

(c) The transmission conditions by considering the fact that integrals of the weak form are additive are obtained by writing the weak form (12) for the two domains split by the interface Γ:

Domain 1:

$$\int_{\Omega_1} (\boldsymbol{\nabla} \times \delta\boldsymbol{u}) \cdot (\nu \, \boldsymbol{\nabla} \times \boldsymbol{u}) - \int_{\partial\Omega_1 \cap \partial\Omega} \delta\boldsymbol{u} \cdot (\nu \, \boldsymbol{\nabla} \times \boldsymbol{u} \times \boldsymbol{n}_1)$$
$$- \int_{\partial\Omega_1 \cap \Gamma} \delta\boldsymbol{u} \cdot (\nu \, \boldsymbol{\nabla} \times \boldsymbol{u} \times \boldsymbol{n}_1) = \int_{\Omega_1} \delta\boldsymbol{u} \cdot \boldsymbol{f} \tag{15}$$

Domain 2:

$$\int_{\Omega_2} (\boldsymbol{\nabla} \times \delta\boldsymbol{u}) \cdot (\nu \, \boldsymbol{\nabla} \times \boldsymbol{u}) - \int_{\partial\Omega_2 \cap \partial\Omega} \delta\boldsymbol{u} \cdot (\nu \, \boldsymbol{\nabla} \times \boldsymbol{u} \times \boldsymbol{n}_2)$$
$$- \int_{\partial\Omega_2 \cap \Gamma} \delta\boldsymbol{u} \cdot (\nu \, \boldsymbol{\nabla} \times \boldsymbol{u} \times \boldsymbol{n}_2) = \int_{\Omega_2} \delta\boldsymbol{u} \cdot \boldsymbol{f} \tag{16}$$

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By adding the two equations (15) and (16) and comparing with the weak form (12) for the full domain Ω, then the extra terms should be equal to zero, this yields:

$$\int_\Gamma \delta\boldsymbol{u} \cdot \nu \left(\boldsymbol{\nabla} \times \boldsymbol{u} \times \boldsymbol{n}_1 + \boldsymbol{\nabla} \times \boldsymbol{u} \times \boldsymbol{n}_2\right) = 0$$

therefore, the second transmission condition is:

$$\boxed{[\![\nu\left(\boldsymbol{\nabla} \times \boldsymbol{u} \times \boldsymbol{n}\right)]\!]_\Gamma = \boldsymbol{0}} \longrightarrow \text{2nd transmission condition - weak continuity}$$

1.3 Question 3: Navier equations for an elastic material

(a) The Navier equations for an elastic material can be written in three different ways:

$$-2\mu \boldsymbol{\nabla} \cdot \bar{\bar{\varepsilon}}(\boldsymbol{u}) - \lambda \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = \rho \boldsymbol{b}$$
$$-\mu \boldsymbol{\Delta} \boldsymbol{u} - (\lambda + \mu) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = \rho \boldsymbol{b}$$
$$\mu \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) - (\lambda + 2\mu) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = \rho \boldsymbol{b}$$

where $\boldsymbol{u}$ is the displacement field, $\bar{\bar{\varepsilon}}(\boldsymbol{u})$ the symmetric part of $\boldsymbol{\nabla}\boldsymbol{u}$, λ and μ the Lamé coefficients, ρ the density of the material and $\boldsymbol{b}$ the body forces. Let us assume that $\boldsymbol{u} = \boldsymbol{0}$ on $\delta\Omega$.

First equation:
To obtain the variational or weak form, pre-multiply by a vector test function $\delta\boldsymbol{u}$ and integrate over the domain Ω:

$$\int_\Omega \delta\boldsymbol{u} \cdot \left(-2\mu \boldsymbol{\nabla} \cdot \bar{\bar{\varepsilon}}(\boldsymbol{u}) \right) - \int_\Omega \delta\boldsymbol{u} \cdot \lambda \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = \int_\Omega \delta\boldsymbol{u} \cdot \rho \boldsymbol{b} \tag{17}$$

For the integration by parts, first recall that:

$$\boldsymbol{\nabla} \cdot (\bar{\bar{\varepsilon}} \, \delta\boldsymbol{u}) = \delta\boldsymbol{u} \cdot (\boldsymbol{\nabla} \cdot \bar{\bar{\varepsilon}}) + \bar{\bar{\varepsilon}} : \boldsymbol{\nabla}\delta\boldsymbol{u}$$
$$\boldsymbol{\nabla} \cdot (\delta\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})) = \delta\boldsymbol{u} \cdot \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + (\boldsymbol{\nabla} \cdot \delta\boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u})$$

Using this information, the integral (17) is written as:

$$\int_\Omega 2\mu \boldsymbol{\nabla}\delta\boldsymbol{u} : \bar{\bar{\varepsilon}} - \int_\Omega 2\mu \boldsymbol{\nabla} \cdot (\bar{\bar{\varepsilon}} \, \delta\boldsymbol{u}) + \int_\Omega \lambda (\boldsymbol{\nabla} \cdot \delta\boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \int_\Omega \lambda \boldsymbol{\nabla} \cdot (\delta\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})) = \int_\Omega \delta\boldsymbol{u} \cdot \rho \boldsymbol{b} \tag{18}$$

Applying divergence theorem to the second and fourth terms on the LHS yields:

$$\int_\Omega 2\mu \boldsymbol{\nabla}\delta\boldsymbol{u} : \bar{\bar{\varepsilon}} - \int_{\partial\Omega} 2\mu (\bar{\bar{\varepsilon}} \, \delta\boldsymbol{u}) \cdot \boldsymbol{n} + \int_\Omega \lambda (\boldsymbol{\nabla} \cdot \delta\boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \int_{\partial\Omega} \lambda (\delta\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})) \cdot \boldsymbol{n} = \int_\Omega \delta\boldsymbol{u} \cdot \rho \boldsymbol{b} \tag{19}$$

Since $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$ (Dirchlet boundary only on $\partial\Omega$, i.e. $\delta\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$), therefore the weak form of the problem is written as:

$$\boxed{\int_\Omega 2\mu \boldsymbol{\nabla}\delta\boldsymbol{u} : \bar{\bar{\varepsilon}} + \int_\Omega \lambda (\boldsymbol{\nabla} \cdot \delta\boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = \int_\Omega \delta\boldsymbol{u} \cdot \rho \boldsymbol{b}} \tag{20}$$

Recalling that $\bar{\bar{\varepsilon}} = \boldsymbol{\nabla}^s u = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T)$, it is seen from the integrals that $\boldsymbol{\nabla}\boldsymbol{u}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{u}$ have to be square integrable. Therefore, $\boldsymbol{u} \in \mathcal{H}^1(\Omega)$ and $\boldsymbol{u} \in \mathcal{H}^{div}(\Omega)$. In fact, $\boldsymbol{u}$ has to be in the less regular space.

Therefore, the space of functions for $\boldsymbol{u}$ and $\delta\boldsymbol{u}$ is:

$$\boxed{\boldsymbol{u} \in \mathcal{H}^1(\Omega) \cap \mathcal{H}^{div}(\Omega) = \mathcal{H}^1(\Omega) \quad \text{such that} \quad \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}}$$

$$\boxed{\delta\boldsymbol{u} \in \mathcal{H}^1(\Omega) \cap \mathcal{H}^{div}(\Omega) = \mathcal{H}^1(\Omega) \quad \text{such that} \quad \delta\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}}$$

where $\mathcal{H}^{div}(\Omega)$ is the space of vector functions of dimension (d) that are defined in Ω such that a function plus its divergence are square integrable, namely:

$$\mathcal{H}^{div}(\Omega) := \left\{ \boldsymbol{u} : \Omega \to \mathbb{R}^d \;\Big|\; \boldsymbol{u} \in \left[\mathcal{L}_2(\Omega)\right]^d, \; \boldsymbol{\nabla} \cdot \boldsymbol{u} \in \mathcal{L}_2(\Omega) \right\}$$

and $\mathcal{H}^1(\Omega)$ is the space of vector functions of dimension (d) that are defined in Ω such that a function plus its gradient are square integrable, namely:

$$\mathcal{H}^1(\Omega) := \left\{ \boldsymbol{u} : \Omega \to \mathbb{R}^d \;\Big|\; \boldsymbol{u} \in \left[\mathcal{L}_2(\Omega)\right]^d, \; \boldsymbol{\nabla}\boldsymbol{u} \in \left[\mathcal{L}_2(\Omega)\right]^{d \times d} \right\}$$

Second equation:

To obtain the variational or weak form, pre-multiply by a vector test function $\delta\boldsymbol{u}$ and integrate over the domain Ω:

$$\int_\Omega \delta\boldsymbol{u} \cdot -\mu\boldsymbol{\Delta}\boldsymbol{u} - \int_\Omega \delta\boldsymbol{u} \cdot (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = \int_\Omega \delta\boldsymbol{u} \cdot \rho\boldsymbol{b} \tag{21}$$

where $\boldsymbol{\Delta}\boldsymbol{u} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\boldsymbol{u}$.

For the integration by parts, first recall that:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{u}\,\delta\boldsymbol{u}) = \delta\boldsymbol{u} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\boldsymbol{u}) + \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\delta\boldsymbol{u} = \delta\boldsymbol{u} \cdot \boldsymbol{\Delta}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\delta\boldsymbol{u}$$
$$\boldsymbol{\nabla}\cdot(\delta\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})) = \delta\boldsymbol{u} \cdot \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + (\boldsymbol{\nabla} \cdot \delta\boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u})$$

Using this information, the integral (21) is written as:

$$\int_\Omega \mu\boldsymbol{\nabla}\delta\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{u} - \int_\Omega \mu\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{u}\,\delta\boldsymbol{u}) + \int_\Omega (\lambda+\mu)(\boldsymbol{\nabla}\cdot\delta\boldsymbol{u})(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \int_\Omega (\lambda+\mu)\boldsymbol{\nabla}\cdot\big(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\big) = \int_\Omega \delta\boldsymbol{u}\cdot\rho\boldsymbol{b} \tag{22}$$

Applying divergence theorem to the second and fourth terms on the LHS yields:

$$\int_\Omega \mu\boldsymbol{\nabla}\delta\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{u} - \int_{\partial\Omega} \mu(\boldsymbol{\nabla}\boldsymbol{u}\,\delta\boldsymbol{u})\cdot\boldsymbol{n} + \int_\Omega (\lambda+\mu)(\boldsymbol{\nabla}\cdot\delta\boldsymbol{u})(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \int_{\partial\Omega} (\lambda+\mu)\big(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\big)\cdot\boldsymbol{n} = \int_\Omega \delta\boldsymbol{u}\cdot\rho\boldsymbol{b} \tag{23}$$

Since $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$ (Dirchlet boundary only on $\partial\Omega$, i.e. $\delta\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$), therefore the weak form of the problem is written as:

$$\boxed{\int_\Omega \mu\boldsymbol{\nabla}\delta\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{u} + \int_\Omega (\lambda + \mu)(\boldsymbol{\nabla} \cdot \delta\boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = \int_\Omega \delta\boldsymbol{u} \cdot \rho\boldsymbol{b}} \tag{24}$$

It is seen from the integrals that $\boldsymbol{\nabla}\boldsymbol{u}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{u}$ have to be square integrable. Therefore, $\boldsymbol{u} \in \mathcal{H}^1(\Omega)$ and $\boldsymbol{u} \in \mathcal{H}^{div}(\Omega)$. In fact, $\boldsymbol{u}$ has to be in the less regular space.

Therefore, the space of functions for $\boldsymbol{u}$ and $\delta\boldsymbol{u}$ is:

$$\boxed{\boldsymbol{u} \in \mathcal{H}^1(\Omega) \cap \mathcal{H}^{div}(\Omega) = \mathcal{H}^1(\Omega) \quad \text{such that } \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}}$$

$$\boxed{\delta\boldsymbol{u} \in \mathcal{H}^1(\Omega) \cap \mathcal{H}^{div}(\Omega) = \mathcal{H}^1(\Omega) \quad \text{such that } \delta\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}}$$

where $\mathcal{H}^{div}(\Omega)$ is the space of vector functions of dimension (d) that are defined in Ω such that a function plus its divergence are square integrable, namely:

$$\mathcal{H}^{div}(\Omega) := \left\{ \boldsymbol{u} : \Omega \to \mathbb{R}^d \;\middle|\; \boldsymbol{u} \in \left[\mathcal{L}_2(\Omega)\right]^d, \; \boldsymbol{\nabla} \cdot \boldsymbol{u} \in \mathcal{L}_2(\Omega) \right\}$$

and $\mathcal{H}^1(\Omega)$ is the space of vector functions of dimension (d) that are defined in Ω such that a function plus its gradient are square integrable, namely:

$$\mathcal{H}^1(\Omega) := \left\{ \boldsymbol{u} : \Omega \to \mathbb{R}^d \;\middle|\; \boldsymbol{u} \in \left[\mathcal{L}_2(\Omega)\right]^d, \; \boldsymbol{\nabla}\boldsymbol{u} \in \left[\mathcal{L}_2(\Omega)\right]^{d \times d} \right\}$$

Third equation:
To obtain the variational or weak form, pre-multiply by a vector test function $\delta\boldsymbol{u}$ and integrate over the domain Ω:

$$\int_\Omega \delta\boldsymbol{u} \cdot (\mu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u}) - \int_\Omega \delta\boldsymbol{u} \cdot (\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = \int_\Omega \delta\boldsymbol{u} \cdot \rho\boldsymbol{b} \tag{25}$$

For the integration by parts, first recall the proven definition given by (10):

$$\delta\boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u}) = (\boldsymbol{\nabla} \times \delta\boldsymbol{u}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) - \boldsymbol{\nabla} \cdot (\delta\boldsymbol{u} \times \boldsymbol{\nabla} \times \boldsymbol{u})$$

and also recall that:

$$\boldsymbol{\nabla} \cdot (\delta\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})) = \delta\boldsymbol{u} \cdot \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + (\boldsymbol{\nabla} \cdot \delta\boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u})$$

Therefore, using this information, the weak form given by (25) is written as:

$$\int_\Omega (\boldsymbol{\nabla} \times \delta\boldsymbol{u}) \cdot (\mu \; \boldsymbol{\nabla} \times \boldsymbol{u}) - \int_\Omega \boldsymbol{\nabla} \cdot (\mu \; \delta\boldsymbol{u} \times \boldsymbol{\nabla} \times \boldsymbol{u}) + \int_\Omega (\lambda + 2\mu)(\boldsymbol{\nabla} \cdot \delta\boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u})$$
$$- \int_\Omega (\lambda + 2\mu)\boldsymbol{\nabla} \cdot \big(\delta\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\big) = \int_\Omega \delta\boldsymbol{u} \cdot \rho\boldsymbol{b}$$

Applying divergence theorem to the second and fourth terms on the LHS yields:

$$\int_\Omega (\boldsymbol{\nabla} \times \delta\boldsymbol{u}) \cdot (\mu \; \boldsymbol{\nabla} \times \boldsymbol{u}) - \int_{\partial\Omega} (\mu \; \boldsymbol{\nabla} \times \boldsymbol{u}) \cdot (\boldsymbol{n} \times \delta\boldsymbol{u}) + \int_\Omega (\lambda + 2\mu)(\boldsymbol{\nabla} \cdot \delta\boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u})$$
$$- \int_{\partial\Omega} (\lambda + 2\mu)\big(\delta\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\big) \cdot \boldsymbol{n} = \int_\Omega \delta\boldsymbol{u} \cdot \rho\boldsymbol{b} \tag{26}$$

Since $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$ (Dirchlet boundary only on $\partial\Omega$, i.e. $\delta\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$), therefore the weak form of the problem is written as:

$$\boxed{\int_\Omega (\boldsymbol{\nabla} \times \delta\boldsymbol{u}) \cdot (\mu \; \boldsymbol{\nabla} \times \boldsymbol{u}) + \int_\Omega (\lambda + 2\mu)(\boldsymbol{\nabla} \cdot \delta\boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = \int_\Omega \delta\boldsymbol{u} \cdot \rho\boldsymbol{b}} \tag{27}$$

It is seen from the integrals that $\boldsymbol{\nabla} \times \boldsymbol{u}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{u}$ have to be square integrable. Therefore, $\boldsymbol{u} \in \mathcal{H}^{curl}(\Omega)$ and $\boldsymbol{u} \in \mathcal{H}^{div}(\Omega)$. In fact, $\boldsymbol{u}$ has to be in the less regular space.

Therefore, the space of functions for $\boldsymbol{u}$ and $\delta\boldsymbol{u}$ is:

$$\boxed{\boldsymbol{u} \in \mathcal{H}^{curl}(\Omega) \cap \mathcal{H}^{div}(\Omega) \quad \text{such that} \quad \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}}$$

$$\boxed{\delta\boldsymbol{u} \in \mathcal{H}^{curl}(\Omega) \cap \mathcal{H}^{div}(\Omega) \quad \text{such that} \quad \delta\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}}$$

where $\mathcal{H}^{div}(\Omega)$ is the space of vector functions of dimension (d) that are defined in Ω such that a function plus its divergence are square integrable, namely:

$$\mathcal{H}^{div}(\Omega) := \left\{ \boldsymbol{u} : \Omega \to \mathbb{R}^d \;\middle|\; \boldsymbol{u} \in \left[\mathcal{L}_2(\Omega)\right]^d, \; \boldsymbol{\nabla} \cdot \boldsymbol{u} \in \mathcal{L}_2(\Omega) \right\}$$

and $\mathcal{H}^{curl}(\Omega)$ is the space of vector functions of dimension (d) that are defined in Ω such that a function plus its curl are square integrable, namely:

$$\mathcal{H}^{curl}(\Omega) := \left\{ \boldsymbol{u} : \Omega \to \mathbb{R}^d \;\middle|\; \boldsymbol{u} \in \left[\mathcal{L}_2(\Omega)\right]^d, \; \boldsymbol{\nabla} \times \boldsymbol{u} \in \left[\mathcal{L}_2(\Omega)\right]^d \right\}$$

(b) **First equation:**

The transmission conditions by considering the fact that integrals of the weak form are additive are obtained by writing the weak form (19) for the two domains Ω_1 and Ω_2 split by the interface Γ:

Domain 1:

$$\int_{\Omega_1} 2\mu\boldsymbol{\nabla}\delta\boldsymbol{u} : \bar{\bar{\varepsilon}} - \int_{\partial\Omega_1\cap\partial\Omega} 2\mu(\bar{\bar{\varepsilon}}\,\delta\boldsymbol{u})\cdot\boldsymbol{n}_1 - \int_{\partial\Omega_1\cap\Gamma} 2\mu(\bar{\bar{\varepsilon}}\,\delta\boldsymbol{u})\cdot\boldsymbol{n}_1 + \int_{\Omega_1} \lambda(\boldsymbol{\nabla}\cdot\delta\boldsymbol{u})(\boldsymbol{\nabla}\cdot\boldsymbol{u})$$
$$- \int_{\partial\Omega_1\cap\partial\Omega} \lambda\big(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\big)\cdot\boldsymbol{n}_1 - \int_{\partial\Omega_1\cap\Gamma} \lambda\big(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\big)\cdot\boldsymbol{n}_1 = \int_{\Omega_1} \delta\boldsymbol{u}\cdot\rho\boldsymbol{b} \tag{28}$$

Domain 2:

$$\int_{\Omega_2} 2\mu\boldsymbol{\nabla}\delta\boldsymbol{u} : \bar{\bar{\varepsilon}} - \int_{\partial\Omega_2\cap\partial\Omega} 2\mu(\bar{\bar{\varepsilon}}\,\delta\boldsymbol{u})\cdot\boldsymbol{n}_2 - \int_{\partial\Omega_2\cap\Gamma} 2\mu(\bar{\bar{\varepsilon}}\,\delta\boldsymbol{u})\cdot\boldsymbol{n}_2 + \int_{\Omega_2} \lambda(\boldsymbol{\nabla}\cdot\delta\boldsymbol{u})(\boldsymbol{\nabla}\cdot\boldsymbol{u})$$
$$- \int_{\partial\Omega_2\cap\partial\Omega} \lambda\big(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\big)\cdot\boldsymbol{n}_2 - \int_{\partial\Omega_2\cap\Gamma} \lambda\big(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\big)\cdot\boldsymbol{n}_2 = \int_{\Omega_2} \delta\boldsymbol{u}\cdot\rho\boldsymbol{b} \tag{29}$$

By adding the two equations (28) and (29) and comparing with the weak form (19) for the full domain Ω, then the extra terms should be equal to zero, this yields:

$$\int_{\Gamma} 2\big[\mu(\bar{\bar{\varepsilon}}\,\delta\boldsymbol{u})\cdot\boldsymbol{n}_1 + \mu(\bar{\bar{\varepsilon}}\,\delta\boldsymbol{u})\cdot\boldsymbol{n}_2\big] = 0 \xrightarrow{\bar{\bar{\varepsilon}}=\bar{\bar{\varepsilon}}^T} \int_{\Gamma} 2\big[\mu\,\bar{\bar{\varepsilon}}\,\boldsymbol{n}_1 + \mu\,\bar{\bar{\varepsilon}}\,\boldsymbol{n}_2\big]\cdot\delta\boldsymbol{u}$$

$$\longrightarrow \boxed{[\![\mu\,\bar{\bar{\varepsilon}}\,\boldsymbol{n}]\!]_{\Gamma} = \boldsymbol{0}} \longrightarrow \text{weak continuity}$$

$$\int_{\Gamma} \big[\lambda\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\cdot\boldsymbol{n}_1 + \lambda\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\cdot\boldsymbol{n}_2\big] = 0 \longrightarrow \int_{\Gamma} \big[\lambda(\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{n}_1 + \lambda(\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{n}_2\big]\cdot\delta\boldsymbol{u}$$

$$\longrightarrow \boxed{[\![\lambda(\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{n}]\!]_{\Gamma} = \boldsymbol{0}} \longrightarrow \text{weak continuity}$$

Second equation:
The transmission conditions by considering the fact that integrals of the weak form are additive are obtained by writing the weak form (23) for the two domains Ω_1 and Ω_2 split by the interface Γ:

Domain 1:

$$\int_{\Omega_1} \mu \boldsymbol{\nabla} \delta \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{u} \; - \int_{\partial\Omega_1 \cap \partial\Omega} \mu(\boldsymbol{\nabla} \boldsymbol{u} \; \delta \boldsymbol{u}) \cdot \boldsymbol{n}_1 - \int_{\partial\Omega_1 \cap \Gamma} \mu(\boldsymbol{\nabla} \boldsymbol{u} \; \delta \boldsymbol{u}) \cdot \boldsymbol{n}_1 + \int_{\Omega_1} (\lambda + \mu)(\boldsymbol{\nabla} \cdot \delta \boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u})$$
$$- \int_{\partial\Omega_1 \cap \partial\Omega} (\lambda + \mu)\big(\delta \boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\big) \cdot \boldsymbol{n}_1 - \int_{\partial\Omega_1 \cap \Gamma} (\lambda + \mu)\big(\delta \boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\big) \cdot \boldsymbol{n}_1 = \int_{\Omega_1} \delta \boldsymbol{u} \cdot \rho \boldsymbol{b}$$
$$(30)$$

Domain 2:

$$\int_{\Omega_2} \mu \boldsymbol{\nabla} \delta \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{u} \; - \int_{\partial\Omega_2 \cap \partial\Omega} \mu(\boldsymbol{\nabla} \boldsymbol{u} \; \delta \boldsymbol{u}) \cdot \boldsymbol{n}_2 - \int_{\partial\Omega_2 \cap \Gamma} \mu(\boldsymbol{\nabla} \boldsymbol{u} \; \delta \boldsymbol{u}) \cdot \boldsymbol{n}_2 + \int_{\Omega_2} (\lambda + \mu)(\boldsymbol{\nabla} \cdot \delta \boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u})$$
$$- \int_{\partial\Omega_2 \cap \partial\Omega} (\lambda + \mu)\big(\delta \boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\big) \cdot \boldsymbol{n}_2 - \int_{\partial\Omega_2 \cap \Gamma} (\lambda + \mu)\big(\delta \boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\big) \cdot \boldsymbol{n}_2 = \int_{\Omega_2} \delta \boldsymbol{u} \cdot \rho \boldsymbol{b}$$
$$(31)$$

By adding the two equations (30) and (31) and comparing with the weak form (23) for the full domain Ω, then the extra terms should be equal to zero, this yields:

$$\int_{\Gamma} \big[\mu(\boldsymbol{\nabla} \boldsymbol{u} \; \delta \boldsymbol{u}) \cdot \boldsymbol{n}_1 + \mu(\boldsymbol{\nabla} \boldsymbol{u} \; \delta \boldsymbol{u}) \cdot \boldsymbol{n}_2 \big] = 0 \longrightarrow \int_{\Gamma} \big[\mu(\boldsymbol{\nabla} \boldsymbol{u})^T \boldsymbol{n}_1 + \mu(\boldsymbol{\nabla} \boldsymbol{u})^T \boldsymbol{n}_2 \big] \cdot \delta \boldsymbol{u}$$
$$\longrightarrow \boxed{[\![\mu(\boldsymbol{\nabla} \boldsymbol{u})^T \boldsymbol{n}]\!]_{\Gamma} = \boldsymbol{0}} \longrightarrow \text{weak continuity}$$
$$\int_{\Gamma} \big[(\lambda + \mu)\delta \boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) \cdot \boldsymbol{n}_1 + (\lambda + \mu)\delta \boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) \cdot \boldsymbol{n}_2 \big] = 0$$
$$\longrightarrow \int_{\Gamma} \big[(\lambda + \mu)(\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{n}_1 + (\lambda + \mu)(\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{n}_2 \big] \cdot \delta \boldsymbol{u}$$
$$\longrightarrow \boxed{[\![(\lambda + \mu)(\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{n}]\!]_{\Gamma} = \boldsymbol{0}} \longrightarrow \text{weak continuity}$$

Third equation:
The transmission conditions by considering the fact that integrals of the weak form are additive are obtained by writing the weak form (26) for the two domains Ω_1 and Ω_2 split by the interface Γ:

Domain 1:

$$\int_{\Omega_1} (\boldsymbol{\nabla} \times \delta \boldsymbol{u}) \cdot (\mu \; \boldsymbol{\nabla} \times \boldsymbol{u}) - \int_{\partial\Omega_1 \cap \partial\Omega} (\mu \; \boldsymbol{\nabla} \times \boldsymbol{u}) \cdot (\boldsymbol{n}_1 \times \delta \boldsymbol{u}) - \int_{\partial\Omega_1 \cap \Gamma} (\mu \; \boldsymbol{\nabla} \times \boldsymbol{u}) \cdot (\boldsymbol{n}_1 \times \delta \boldsymbol{u})$$
$$+ \int_{\Omega_1} (\lambda + 2\mu)(\boldsymbol{\nabla} \cdot \delta \boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \int_{\partial\Omega_1 \cap \partial\Omega} (\lambda + 2\mu)\big(\delta \boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\big) \cdot \boldsymbol{n}_1$$
$$- \int_{\partial\Omega_1 \cap \Gamma} (\lambda + 2\mu)\big(\delta \boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\big) \cdot \boldsymbol{n}_1 = \int_{\Omega_1} \delta \boldsymbol{u} \cdot \rho \boldsymbol{b}$$
$$(32)$$

Domain 2:

$$\int_{\Omega_2} (\boldsymbol{\nabla} \times \delta\boldsymbol{u}) \cdot (\mu\, \boldsymbol{\nabla} \times \boldsymbol{u}) - \int_{\partial\Omega_2 \cap \partial\Omega} (\mu\, \boldsymbol{\nabla} \times \boldsymbol{u}) \cdot (\boldsymbol{n}_2 \times \delta\boldsymbol{u}) - \int_{\partial\Omega_2 \cap \Gamma} (\mu\, \boldsymbol{\nabla} \times \boldsymbol{u}) \cdot (\boldsymbol{n}_2 \times \delta\boldsymbol{u})$$

$$+ \int_{\Omega_2} (\lambda + 2\mu)(\boldsymbol{\nabla} \cdot \delta\boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \int_{\partial\Omega_2 \cap \partial\Omega} (\lambda + 2\mu)\big(\delta\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\big) \cdot \boldsymbol{n}_2$$

$$- \int_{\partial\Omega_2 \cap \Gamma} (\lambda + 2\mu)\big(\delta\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\big) \cdot \boldsymbol{n}_2 = \int_{\Omega_2} \delta\boldsymbol{u} \cdot \rho\boldsymbol{b}$$

$$(33)$$

By adding the two equations (32) and (33) and comparing with the weak form (26) for the full domain Ω, then the extra terms should be equal to zero, this yields:

$$\int_\Gamma \big[(\mu\, \boldsymbol{\nabla} \times \boldsymbol{u}) \cdot (\boldsymbol{n}_1 \times \delta\boldsymbol{u}) + (\mu\, \boldsymbol{\nabla} \times \boldsymbol{u}) \cdot (\boldsymbol{n}_2 \times \delta\boldsymbol{u})\big]$$

$$\longrightarrow \int_\Gamma \delta\boldsymbol{u} \cdot (\mu\boldsymbol{\nabla} \times \boldsymbol{u} \times \boldsymbol{n}_1 + \mu\boldsymbol{\nabla} \times \boldsymbol{u} \times \boldsymbol{n}_2) = 0$$

$$\longrightarrow \boxed{[\![\mu\boldsymbol{\nabla} \times \boldsymbol{u} \times \boldsymbol{n}]\!]_\Gamma = \boldsymbol{0}} \longrightarrow \text{weak continuity}$$

$$\int_\Gamma \big[(\lambda + 2\mu)\delta\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) \cdot \boldsymbol{n}_1 + (\lambda + 2\mu)\delta\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) \cdot \boldsymbol{n}_2\big] = 0$$

$$\longrightarrow \int_\Gamma \big[(\lambda + 2\mu)(\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{n}_1 + (\lambda + 2\mu)(\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{n}_2\big] \cdot \delta\boldsymbol{u}$$

$$\longrightarrow \boxed{[\![(\lambda + 2\mu)(\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{n}]\!]_\Gamma = \boldsymbol{0}} \longrightarrow \text{weak continuity}$$

2 Coupling in space of homogeneous problems: Domain decomposition methods

2.1 Question 1: Euler-Bernoulli beam

Considering the beam problem shown below, where the subdomains Ω_1 and Ω_2 and the overlapping Ω_{12} are defined:

$$\Omega_1 = [0, L_1]$$
$$\Omega_2 = [L_2, L]$$
$$\Omega_{12} = \Omega_1 \cap \Omega_2 = [L_2, L_1]$$
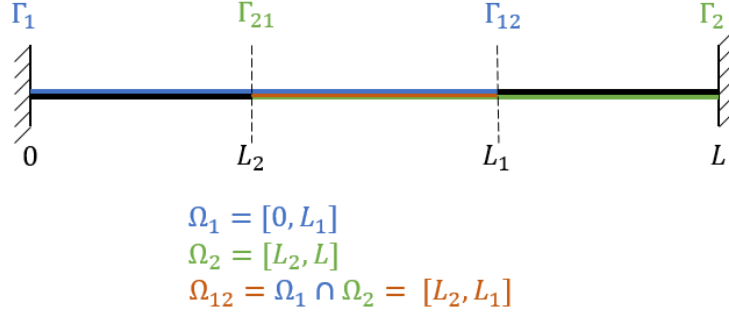
and defining an operator $\mathcal{T} := EI \dfrac{d^4}{dx^4}$

(a) The iteration-by-subdomain scheme based on a Schwarz additive Domain Decomposition Method (DDM) is written as:

$$
\begin{aligned}
\mathcal{T} v_1^{(k)} &= f && \text{in } \Omega_1 \\
v_1^{(k)} &= 0 && \text{on } \Gamma_1 \\
\frac{dv_1^{(k)}}{dx} &= 0 && \text{on } \Gamma_1 \\
v_1^{(k)} &= v_2^{(k-1)} && \text{on } \Gamma_{12} \\
\frac{dv_1^{(k)}}{dx} &= \frac{dv_2^{(k-1)}}{dx} && \text{on } \Gamma_{12}
\end{aligned}
\qquad
\begin{aligned}
\mathcal{T} v_2^{(k)} &= f && \text{in } \Omega_2 \\
v_2^{(k)} &= 0 && \text{on } \Gamma_2 \\
\frac{dv_2^{(k)}}{dx} &= 0 && \text{on } \Gamma_2 \\
v_2^{(k)} &= v_1^{(k-1)} && \text{on } \Gamma_{21} \\
\frac{dv_2^{(k)}}{dx} &= \frac{dv_1^{(k-1)}}{dx} && \text{on } \Gamma_{21}
\end{aligned}
$$

(b) To obtain the matrix form using finite element discretization, first the domain Ω is divided into finite element partitions $(\Omega = \cup \, \mathcal{K})$.

The solutions of the displacement v and rotation θ are approximated as:

$$v \approx v_h, \text{ such that } v_h|_{\mathcal{K}} \in \mathcal{P}_p(\mathcal{K})$$

$$\theta = \frac{dv}{dx} \approx \theta_h, \text{ such that } \theta_h|_{\mathcal{K}} \in \mathcal{P}_p(\mathcal{K})$$

where $\mathcal{P}_p$ is the polynomial space of order p.

Furthermore, $v_h \in \mathcal{C}^1(\Omega)$

$$
\left.
\begin{array}{l}
v_h|_{\mathcal{K}} \text{ is a polynomial} \\
v_h \in \mathcal{H}^2(\Omega)
\end{array}
\right\} \Rightarrow v_h \in \mathcal{C}^1(\Omega)
$$

To achieve $\mathcal{C}^1$-continuity, Hermite shape functions are used in the finite element approximation [1]. For this, we define the scalar variable u which is used to approximate both the displacement v and rotation θ.

Considering the Hermite interpolation:

$$u(x) \approx u_h(x) = \sum_a (N^a(x)\ V^a + M^a(x)\ \Theta^a) = \{N^1\ M^1\ ...\ N^n\ M^n\} \begin{Bmatrix} V^1 \\ \Theta^1 \\ \vdots \\ V^n \\ \Theta^n \end{Bmatrix} = \boldsymbol{N}\boldsymbol{v}$$

$$\delta u(x) \approx \delta u_h(x) = \sum_b (N^b(x)\ \delta V^b + M^b(x)\ \delta\Theta^b) = \{\delta V^1\ \delta\Theta^1\ ...\ \delta V^n\ \delta\Theta^n\} \begin{Bmatrix} N^1 \\ M^1 \\ \vdots \\ N^n \\ M^n \end{Bmatrix} = \boldsymbol{\delta v}\ \boldsymbol{N}^T$$

where $N^a(x)$ and $M^a(x)$ are the Hermite shape functions defined in [1], for $a = 1, ..., n$. The nodal values V^a, Θ^a, δV^b, and $\delta\Theta^b$ are defined as:

$$V^a = v_h(x^a), \quad \Theta^a = \theta_h(x^a), \quad \delta V^b = \delta v_h(x^b), \quad \delta\Theta^b = \delta\theta_h(x^b)$$

The discrete form of the problem is then written as:

$$EI \int_0^L \frac{d^2\delta u_h}{dx^2}\frac{d^2 u_h}{dx^2} = \int_0^L \delta u_h\ f$$

Substituting the Hermite interpolation yields:

$$EI \int_0^L \frac{d^2\boldsymbol{N}^T}{dx^2}\frac{d^2\boldsymbol{N}}{dx^2}\ \boldsymbol{v} = \int_0^L \delta\boldsymbol{N}^T\ f$$

That is denoted by:

$$\boxed{\boldsymbol{B}\boldsymbol{v} = \boldsymbol{f}}$$

Finally, the matrix version of the iteration-by-subdomain scheme based on a Schwarz additive Domain Decomposition Method (DDM) considering the mesh shown in 2 is written as:

$\boldsymbol{B}_1\boldsymbol{v}_1^{(k)} = \boldsymbol{f}_1$	in Ω_1
$\boldsymbol{v}_1(1)^{(k)} = 0$	on Γ_1
$\boldsymbol{v}_1(2)^{(k)} = 0$	on Γ_1
$\boldsymbol{v}_1(2j-1)^{(k)} = \boldsymbol{v}_2(2(j-i)+1)^{(k-1)}$	on Γ_{12}
$\boldsymbol{v}_1(2j)^{(k)} = \boldsymbol{v}_2(2(j-i)+2)^{(k-1)}$	on Γ_{12}

$\boldsymbol{B}_2\boldsymbol{v}_2^{(k)} = \boldsymbol{f}_2$	in Ω_2
$\boldsymbol{v}_2(2(n-i)+1)^{(k)} = 0$	on Γ_2
$\boldsymbol{v}_2(2(n-i)+2)^{(k)} = 0$	on Γ_2
$\boldsymbol{v}_2(1)^{(k)} = \boldsymbol{v}_1(2i-1)^{(k-1)}$	on Γ_{21}
$\boldsymbol{v}_2(2)^{(k)} = \boldsymbol{v}_1(2i)^{(k-1)}$	on Γ_{21}

where the vector $\boldsymbol{v}_1$ contains the nodal values associated to the nodes of the domain Ω_1, and it is of size $2j$. The vector $\boldsymbol{v}_2$ contains the nodal values associated to the nodes of the domain Ω_2, and it is of size $2(n-i)+2$.
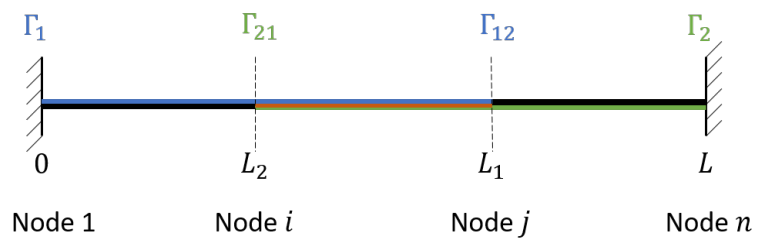
Figure 2: Nodes at the boundaries and interfaces

2.2 Question 2: Maxwell problem

Recalling the Maxwell problem which consists in finding a vector field $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^3$ such that

$$\nu \, \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega$$
$$\boldsymbol{n} \times \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega$$

where $\nu > 0$, $\boldsymbol{f}$ is a divergence free force field and $\boldsymbol{n}$ the unit external normal. Equation $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ is in fact redundant.

First, an operator $\mathcal{M}$ which is applied to a vector $\boldsymbol{u}$ for the Maxwell problem is defined as:

$$\mathcal{M}\boldsymbol{u} := \nu \, \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u}$$

Furthermore, the arbitrary domain Ω is divided into two non-overlapping subdomains Ω_1 and Ω_2 with boundaries $\partial\Omega_1$ and $\partial\Omega_2$, respectively. The interface Γ and the boundaries Γ_1 and Γ_2 are defined as:

$$\Gamma := \partial\Omega_1 \cap \partial\Omega_2$$
$$\Gamma_1 := \partial\Omega \cap \partial\Omega_1$$
$$\Gamma_2 := \partial\Omega \cap \partial\Omega_2$$

Second, if $\boldsymbol{u}_i := \boldsymbol{u}|_{\Omega_i}$, then the Maxwell problem is written for each subdomain Ω_i for $i = 1, 2$:

$$\mathcal{M}\boldsymbol{u}_i = \boldsymbol{f} \quad \text{in } \Omega_i$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_i = 0 \quad \text{in } \Omega_i$$
$$\boldsymbol{n} \times \boldsymbol{u}_i = \boldsymbol{0} \quad \text{on } \Gamma_i$$
$$[\![\boldsymbol{n} \times \boldsymbol{u}]\!] = \boldsymbol{0} \quad \text{on } \Gamma \longrightarrow \text{1st transmission condition - strong continuity}$$
$$[\![\nu \, (\boldsymbol{\nabla} \times \boldsymbol{u} \times \boldsymbol{n})]\!] = \boldsymbol{0} \quad \text{on } \Gamma \longrightarrow \text{2nd transmission condition - weak continuity}$$

(a) Next, the iteration-by-subdomain scheme based on the Dirchlet-Neumann coupling is written as:

$\mathcal{M}\boldsymbol{u}_1^{(k)} = \boldsymbol{f}$ $\qquad$ in Ω_1	$\mathcal{M}\boldsymbol{u}_2^{(k)} = \boldsymbol{f}$ $\qquad$ in Ω_2
$\boldsymbol{\nabla} \cdot \boldsymbol{u}_1^{(k)} = 0$ $\qquad$ in Ω_1	$\boldsymbol{\nabla} \cdot \boldsymbol{u}_2^{(k)} = 0$ $\qquad$ in Ω_2
$\boldsymbol{n} \times \boldsymbol{u}_1^{(k)} = \boldsymbol{0}$ $\qquad$ on Γ_1	$\boldsymbol{n} \times \boldsymbol{u}_2^{(k)} = \boldsymbol{0}$ $\qquad$ on Γ_2
$\nu_1 \, (\boldsymbol{\nabla} \times \boldsymbol{u}_1^{(k)} \times \boldsymbol{n}) = \nu_2 \, (\boldsymbol{\nabla} \times \boldsymbol{u}_2^{(k-1)} \times \boldsymbol{n})$ $\quad$ on Γ	$\boldsymbol{n} \times \boldsymbol{u}_2^{(k)} = -\boldsymbol{n} \times \boldsymbol{u}_1^{(l)}$ $\qquad$ on Γ

where $l = k - 1$ yields a Jacobi scheme which allows for parallel solve, while $l = k$ yields a Gauss-Seidel scheme with sequential solve. It has been used that $\boldsymbol{n} = \boldsymbol{n}_1 = -\boldsymbol{n}_2$ on Γ.

(b) To obtain the expression of the Steklov-Poincaré operator of the problem, the variable $\boldsymbol{u}_i$ is defined as:

$$\boldsymbol{u}_i = \boldsymbol{u}_i^o + \tilde{\boldsymbol{u}}_i, \quad \text{for } i = 1, 2$$

Therefore, the Maxwell problem can be split into two parts as follows:

$$
\boxed{
\begin{aligned}
\mathcal{M}\boldsymbol{u}_i^o &= \boldsymbol{f} && \text{in } \Omega_i \\
\boldsymbol{\nabla}\cdot\boldsymbol{u}_i^o &= 0 && \text{in } \Omega_i \\
\boldsymbol{n}\times\boldsymbol{u}_i^o &= \boldsymbol{0} && \text{on } \Gamma_i \\
\boldsymbol{n}\times\boldsymbol{u}_i^o &= \boldsymbol{0} && \text{on } \Gamma
\end{aligned}
}
\qquad
\boxed{
\begin{aligned}
\mathcal{M}\tilde{\boldsymbol{u}}_i &= \boldsymbol{0} && \text{in } \Omega_i \\
\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{u}}_i &= 0 && \text{in } \Omega_i \\
\boldsymbol{n}\times\tilde{\boldsymbol{u}}_i &= \boldsymbol{0} && \text{on } \Gamma_i \\
\boldsymbol{n}\times\tilde{\boldsymbol{u}}_i &= \boldsymbol{\phi} && \text{on } \Gamma
\end{aligned}
}
$$

The first part of the problem (to the left) can be solved independently for $i = 1, 2$ to obtain $\boldsymbol{u}_1^o$ and $\boldsymbol{u}_2^o$.

The problem is now reduced to obtaining $\boldsymbol{\phi}$ such that $\boldsymbol{u}_i = \boldsymbol{u}_i^o + \tilde{\boldsymbol{u}}_i$ is $\boldsymbol{u}|_{\Omega_i}$. For this, we must satisfy the second transmission condition given by

$$
\nu_1\left(\boldsymbol{\nabla}\times\boldsymbol{u}_1\times\boldsymbol{n}\right) = \nu_2\left(\boldsymbol{\nabla}\times\boldsymbol{u}_2\times\boldsymbol{n}\right)
$$

which also reads

$$
\nu_1\left(\boldsymbol{\nabla}\times\tilde{\boldsymbol{u}}_1\times\boldsymbol{n}\right) + \nu_1\left(\boldsymbol{\nabla}\times\boldsymbol{u}_1^o\times\boldsymbol{n}\right) = \nu_2\left(\boldsymbol{\nabla}\times\tilde{\boldsymbol{u}}_2\times\boldsymbol{n}\right) + \nu_2\left(\boldsymbol{\nabla}\times\boldsymbol{u}_2^o\times\boldsymbol{n}\right)
$$

Re-arranging yields

$$
\nu_1\left(\boldsymbol{\nabla}\times\tilde{\boldsymbol{u}}_1\times\boldsymbol{n}\right) - \nu_2\left(\boldsymbol{\nabla}\times\tilde{\boldsymbol{u}}_2\times\boldsymbol{n}\right) = -\nu_1\left(\boldsymbol{\nabla}\times\boldsymbol{u}_1^o\times\boldsymbol{n}\right) + \nu_2\left(\boldsymbol{\nabla}\times\boldsymbol{u}_2^o\times\boldsymbol{n}\right)
$$

where the right-hand-side of the previous equation is known.

Next, the Steklov-Poincaré operator $\mathcal{S}$ and the known value $\mathcal{G}$ are defined as:

$$
\begin{aligned}
\mathcal{S}: \left[\mathcal{H}^{1/2}(\Gamma)\right]^d &\longrightarrow \left[\mathcal{H}^{-1/2}(\Gamma)\right]^d \\
\boldsymbol{\phi} &\longmapsto \nu_1\left(\boldsymbol{\nabla}\times\tilde{\boldsymbol{u}}_1\times\boldsymbol{n}\right) - \nu_2\left(\boldsymbol{\nabla}\times\tilde{\boldsymbol{u}}_2\times\boldsymbol{n}\right)
\end{aligned}
$$

$$
\mathcal{G} = -\nu_1\left(\boldsymbol{\nabla}\times\boldsymbol{u}_1^o\times\boldsymbol{n}\right) + \nu_2\left(\boldsymbol{\nabla}\times\boldsymbol{u}_2^o\times\boldsymbol{n}\right) \quad \in \left[\mathcal{H}^{-1/2}(\Gamma)\right]^d
$$

Finally, the expression of the Steklov-Poincaré operator of the problem is: Find $\boldsymbol{\phi} \in \left[\mathcal{H}^{1/2}(\Gamma)\right]^d$ such that

$$
\boxed{\mathcal{S}\boldsymbol{\phi} = \mathcal{G}}
$$

(c) Recalling the weak form of the problem given earlier by equation (13):

$$
\int_\Omega (\boldsymbol{\nabla}\times\delta\boldsymbol{u})\cdot(\nu\,\boldsymbol{\nabla}\times\boldsymbol{u}) = \int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{f}
$$

The variable $\boldsymbol{u}$ is approximated using polynomial interpolation as:

$$
\boldsymbol{u} = \begin{Bmatrix} u^1 \\ \vdots \\ u^d \end{Bmatrix} \approx \begin{Bmatrix} u_h^1 \\ \vdots \\ u_h^d \end{Bmatrix} = \begin{Bmatrix} \sum_i^n N_i U_i^1 \\ \vdots \\ \sum_i^n N_i U_i^d \end{Bmatrix} = \sum_i^n N_i \begin{Bmatrix} U_i^1 \\ \vdots \\ U_i^d \end{Bmatrix} = \sum_i^n N_i \boldsymbol{U}_i = \sum_i^n \boldsymbol{N}_i \boldsymbol{U}_i
$$

where $N_i(\boldsymbol{x}_j) = \delta_{ij}$ is the shape function associated to the node i, and U_i^j is the i-th nodal value of the component j of the variable $\boldsymbol{u}_h$. It is important to note that the bold $\boldsymbol{N}_i = N_i \boldsymbol{I}$ where $\boldsymbol{I}$ is the identity matrix of dimension d.

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the test function $\delta\boldsymbol{u}$ is approximated in similar manner:

$$\delta\boldsymbol{u} = \begin{Bmatrix} \delta u^1 \\ \vdots \\ \delta u^d \end{Bmatrix} \approx \begin{Bmatrix} \delta u_h^1 \\ \vdots \\ \delta u_h^d \end{Bmatrix} = \begin{Bmatrix} \sum_j^n N_j \delta U_j^1 \\ \vdots \\ \sum_j^n N_j \delta U_j^d \end{Bmatrix} = \sum_j^n N_j \begin{Bmatrix} \delta U_j^1 \\ \vdots \\ \delta U_j^d \end{Bmatrix} = \sum_j^n N_j \delta \boldsymbol{U}_j = \sum_j^n \boldsymbol{N}_j \delta \boldsymbol{U}_j$$

Dropping the coefficients $\delta\boldsymbol{U}_j$ of the test function and substituting the approximations for $\boldsymbol{u}$ and $\delta\boldsymbol{u}$ into the weak form yield the discrete equation for node j as:

$$\int_\Omega (\boldsymbol{\nabla}\times\boldsymbol{N}_j)(\nu\,\boldsymbol{\nabla}\times\boldsymbol{N}_i)\,\boldsymbol{U}_i = \int_\Omega \boldsymbol{N}_j \boldsymbol{f}$$

That is denoted by

$$\boldsymbol{A}_{ji}\boldsymbol{U}_i = \boldsymbol{F}_j \qquad \implies \qquad \boldsymbol{A}\boldsymbol{U} = \boldsymbol{F}$$

Considering the weak forms of the problem in the two subdomain Ω_1 and Ω_2 that were given earlier by equations (15) and (16), and applying the homogeneous Dirchlet boundary conditions strongly on the Γ_1 and Γ_2, the following weak forms are obtained:

Domain 1:

$$\int_{\Omega_1} (\boldsymbol{\nabla}\times\delta\boldsymbol{u})\cdot(\nu_1\,\boldsymbol{\nabla}\times\boldsymbol{u}_1) - \int_{\partial\Omega_1\cap\Gamma} \delta\boldsymbol{u}\cdot(\nu_1\,\boldsymbol{\nabla}\times\boldsymbol{u}_1\times\boldsymbol{n}_1) = \int_{\Omega_1} \delta\boldsymbol{u}\cdot\boldsymbol{f}$$

Domain 2:

$$\int_{\Omega_2} (\boldsymbol{\nabla}\times\delta\boldsymbol{u})\cdot(\nu_2\,\boldsymbol{\nabla}\times\boldsymbol{u}_2) - \int_{\partial\Omega_2\cap\Gamma} \delta\boldsymbol{u}\cdot(\nu_2\,\boldsymbol{\nabla}\times\boldsymbol{u}_2\times\boldsymbol{n}_2) = \int_{\Omega_2} \delta\boldsymbol{u}\cdot\boldsymbol{f}$$

Now, the boundary conditions on the interface Γ are applied for each subdomain (Neumann conditions imposed weakly on $\partial\Omega_1\cap\Gamma$, Dirchlet conditions imposed strongly on $\partial\Omega_2\cap\Gamma$) yielding the following equations:

Domain 1:

$$\int_{\Omega_1} (\boldsymbol{\nabla}\times\delta\boldsymbol{u})\cdot(\nu_1\,\boldsymbol{\nabla}\times\boldsymbol{u}_1) - \int_{\partial\Omega_1\cap\Gamma} \delta\boldsymbol{u}\cdot(\nu_2\,\boldsymbol{\nabla}\times\boldsymbol{u}_2\times\boldsymbol{n}) = \int_{\Omega_1} \delta\boldsymbol{u}\cdot\boldsymbol{f}$$

Domain 2:

$$\int_{\Omega_2} (\boldsymbol{\nabla}\times\delta\boldsymbol{u})\cdot(\nu_2\,\boldsymbol{\nabla}\times\boldsymbol{u}_2) = \int_{\Omega_2} \delta\boldsymbol{u}\cdot\boldsymbol{f}$$

The discrete form can be written by separating the degrees of freedom of the two subdomains Ω_1, Ω_2 and the interface Γ as:

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{1\Gamma} & \boldsymbol{0} \\ \boldsymbol{A}_{\Gamma 1} & \boldsymbol{A}_{\Gamma\Gamma} & \boldsymbol{A}_{\Gamma 2} \\ \boldsymbol{0} & \boldsymbol{A}_{2\Gamma} & \boldsymbol{A}_{22} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_\Gamma \\ \boldsymbol{U}_2 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_\Gamma \\ \boldsymbol{F}_2 \end{Bmatrix}$$

where the sub-indices 1 and 2 are denoting the set of nodes in the subdomains Ω_1 and Ω_2, respectively. Moreover, the sub-index Γ denotes the set of nodes on the interface Γ.

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Manipulating the equations and recalling that $\boldsymbol{A}_{\Gamma\Gamma} = \boldsymbol{A}_{\Gamma\Gamma}^{(1)} + \boldsymbol{A}_{\Gamma\Gamma}^{(2)}$, the iteration-by-subdomain scheme based on the Dirchlet-Neumann coupling is written in matrix form as:

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{1\Gamma} \\ \boldsymbol{A}_{\Gamma 1} & \boldsymbol{A}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U}_1^{(k)} \\ \boldsymbol{U}_\Gamma^{(k)} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_\Gamma \boxed{-\boldsymbol{A}_{\Gamma 2}\boldsymbol{U}_2^{(k-1)} - \boldsymbol{A}_{\Gamma\Gamma}^{(2)}\boldsymbol{U}_\Gamma^{(k-1)}} \end{Bmatrix} \longleftarrow \text{Neumann conditions}$$

$$\boldsymbol{A}_{22}\boldsymbol{U}_2^{(k)} = \boldsymbol{F}_2 \boxed{-\boldsymbol{A}_{2\Gamma}\boldsymbol{U}_\Gamma^{(l)}} \qquad\qquad \longleftarrow \text{Dirchlet conditions}$$

Again, $l = k - 1$ yields a Jacobi scheme which allows for parallel solve, while $l = k$ yields a Gauss-Seidel scheme with sequential solve.

2.3 Question 3: Laplace equation

Considering the problem of finding $u : \Omega \longrightarrow \mathbb{R}$ such that

$$-k\mathbf{\Delta}u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \delta\Omega$$

where $k > 0$. Let Γ be a surface crossing the domain Ω.

Furthermore, the arbitrary domain Ω is divided into two non-overlapping subdomains Ω_1 and Ω_2 with boundaries $\partial\Omega_1$ and $\partial\Omega_2$, respectively. The interface Γ and the boundaries Γ_1 and Γ_2 are defined as:

$$\Gamma := \partial\Omega_1 \cap \partial\Omega_2$$
$$\Gamma_1 := \partial\Omega \cap \partial\Omega_1$$
$$\Gamma_2 := \partial\Omega \cap \partial\Omega_2$$

The Laplace operator $\mathcal{L}$ applied to a scalar variable u is defined as:

$$\mathcal{L}u = -k\mathbf{\Delta}u$$

if $u_i := u|_{\Omega_i}$, then the Laplace problem is written for each subdomain Ω_i for $i = 1, 2$:

$$\mathcal{L}u_i = f \quad \text{in } \Omega_i$$
$$u_i = 0 \quad \text{on } \Gamma_i$$
$$[\![u]\!] = 0 \quad \text{on } \Gamma \longrightarrow \text{1st transmission condition - strong continuity}$$
$$[\![k\frac{\partial u}{\partial n}]\!] = 0 \quad \text{on } \Gamma \longrightarrow \text{2nd transmission condition - weak continuity}$$

(a) Next, the iteration-by-subdomain scheme based on the Dirchlet-Robin coupling is written as:

$$\begin{array}{|ll|}
\hline
\mathcal{L}u_1^{(k)} = f & \text{in } \Omega_1 \\
u_1^{(k)} = 0 & \text{on } \Gamma_1 \\
k_1\dfrac{\partial u_1^{(k)}}{\partial n} + \gamma u_1^{(k)} = k_2\dfrac{\partial u_2^{(k-1)}}{\partial n} + \gamma u_2^{(k-1)} & \text{on } \Gamma \\
\hline
\end{array}
\qquad
\begin{array}{|ll|}
\hline
\mathcal{L}u_2^{(k)} = f & \text{in } \Omega_2 \\
u_2^{(k)} = 0 & \text{on } \Gamma_2 \\
u_2^{(k)} = u_1^{(l)} & \text{on } \Gamma \\
\hline
\end{array}$$

with $\gamma > 0$. Taking $l = k - 1$ yields a Jacobi scheme which allows for parallel solve, while $l = k$ yields a Gauss-Seidel scheme with sequential solve. It has been used that $n = n_1 = -n_2$.

(b) To obtain the weak form of the problem, the strong form is multiplied by a test function $\delta u \in \mathcal{H}^1(\Omega)$ such that $\delta u = 0$ on the the boundary $\partial\Omega$, and integrated over the whole domain:

$$\int_\Omega \delta u(-k\mathbf{\Delta}u) = \int_\Omega \delta u \; f$$

Integrating by parts yields:

$$\int_\Omega \mathbf{\nabla}\delta u \cdot (k\mathbf{\nabla}u) - \int_{\partial\Omega} \delta u(k\frac{\partial u}{\partial n}) = \int_\Omega \delta u \; f$$

and since $\delta u = 0$ on $\partial\Omega$, therefore the weak form is reduced to:

$$\int_\Omega \boldsymbol{\nabla}\delta u \cdot (k\boldsymbol{\nabla} u) = \int_\Omega \delta u\ f$$

The variable u is approximated using polynomial interpolation as:

$$u \approx u_h = \sum_i^n N_i U_i$$

where $N_i(\boldsymbol{x}_j) = \delta_{ij}$ is the shape function associated to the node i, U_i is the nodal value of the variable u_h at node i.

The test function is approximated in a similar manner:

$$\delta u \approx \delta u_h = \sum_j^n N_j \delta U_j$$

Dropping the coefficients δU_j of the test function and substituting the approximations for u and δu into the weak form yield the discrete equation for node j as:

$$\int_\Omega \boldsymbol{\nabla} N_j \cdot (k\boldsymbol{\nabla} N_i)\ U_i = \int_\Omega N_j\ f$$

That is denoted by

$$A_{ji}U_i = F_j \qquad \Longrightarrow \qquad \boldsymbol{A}\boldsymbol{U} = \boldsymbol{F}$$

Considering the weak forms of the problem in the two subdomain Ω_1 and Ω_2 and applying the homogeneous Dirchlet boundary conditions strongly on the Γ_1 and Γ_2, the following weak forms are obtained:

Domain 1:

$$\int_{\Omega_1} \boldsymbol{\nabla}\delta u \cdot (k_1 \boldsymbol{\nabla} u_1) - \int_{\partial\Omega_1 \cap \Gamma} \delta u(k_1 \frac{\partial u_1}{\partial n_1}) = \int_{\Omega_1} \delta u\ f$$

Domain 2:

$$\int_{\Omega_2} \boldsymbol{\nabla}\delta u \cdot (k_2 \boldsymbol{\nabla} u_2) - \int_{\partial\Omega_2 \cap \Gamma} \delta u(k_2 \frac{\partial u_2}{\partial n_2}) = \int_{\Omega_2} \delta u\ f$$

Now, the boundary conditions on the interface Γ are applied for each subdomain (Robin conditions imposed weakly on $\partial\Omega_1 \cap \Gamma$ instead of the Neumann condition appearing in the previous equation, Dirchlet conditions imposed strongly on $\partial\Omega_2 \cap \Gamma$) yielding the following equations:

Domain 1:

$$\int_{\Omega_1} \boldsymbol{\nabla}\delta u \cdot (k_1 \boldsymbol{\nabla} u_1) - \int_{\partial\Omega_1 \cap \Gamma} \delta u(k_2 \frac{\partial u_2}{\partial n} + \gamma u_2) = \int_{\Omega_1} \delta u\ f$$

Domain 2:

$$\int_{\Omega_2} \boldsymbol{\nabla}\delta u \cdot (k_2 \boldsymbol{\nabla} u_2) = \int_{\Omega_2} \delta u\ f$$

The discrete form can be written by separating the degrees of freedom of the two subdomains Ω_1, Ω_2 and the interface Γ as:

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{1\Gamma} & \boldsymbol{0} \\ \boldsymbol{A}_{\Gamma 1} & \boldsymbol{A}_{\Gamma\Gamma} & \boldsymbol{A}_{\Gamma 2} \\ \boldsymbol{0} & \boldsymbol{A}_{2\Gamma} & \boldsymbol{A}_{22} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_\Gamma \\ \boldsymbol{U}_2 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_\Gamma \\ \boldsymbol{F}_2 \end{Bmatrix} \tag{34}$$

where the sub-indices 1 and 2 are denoting the set of nodes in the subdomains Ω_1 and Ω_2, respectively. Moreover, the sub-index Γ denotes the set of nodes on the interface Γ.

Manipulating the equations and recalling that $\boldsymbol{A}_{\Gamma\Gamma} = \boldsymbol{A}_{\Gamma\Gamma}^{(1)} + \boldsymbol{A}_{\Gamma\Gamma}^{(2)}$, the iteration-by-subdomain scheme based on the Dirchlet-Robin coupling is written in matrix form as:

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{1\Gamma} \\ \boldsymbol{A}_{\Gamma 1} & \boldsymbol{A}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U}_1^{(k)} \\ \boldsymbol{U}_\Gamma^{(k)} \end{Bmatrix} = \left\{ \boldsymbol{F}_\Gamma \boxed{-\boldsymbol{A}_{\Gamma 2}\boldsymbol{U}_2^{(k-1)} - \boldsymbol{A}_{\Gamma\Gamma}^{(2)}\boldsymbol{U}_\Gamma^{(k-1)}} \right\} \qquad \longleftarrow \text{ Robin conditions}$$

$$\boldsymbol{A}_{22}\boldsymbol{U}_2^{(k)} = \boldsymbol{F}_2 \boxed{-\boldsymbol{A}_{2\Gamma}\boldsymbol{U}_\Gamma^{(l)}} \qquad \longleftarrow \text{ Dirchlet conditions}$$

$$\tag{35}$$

Again, $l = k - 1$ yields a Jacobi scheme which allows for parallel solve, while $l = k$ yields a Gauss-Seidel scheme with sequential solve.

(c) To obtain the Steklov-Poincaré operator $\mathcal{S}$, the problem at hand is reduced to finding a variable ϕ that satisfies the Robin condition at the interface Γ. After some manipulation similar to what was done in the previous question, the Steklov-Poincaré operator $\mathcal{S}$ and the known value $\mathcal{G}$ are defined as:

$$\begin{aligned} \mathcal{S} : \mathcal{H}^{1/2}(\Gamma) &\longrightarrow \mathcal{H}^{-1/2}(\Gamma) \\ \phi &\longmapsto k_1 \frac{\partial \tilde{u}_1}{\partial n} + \gamma \tilde{u}_1 - k_2 \frac{\partial \tilde{u}_2}{\partial n} - \gamma \tilde{u}_2 \end{aligned}$$

$$\mathcal{G} = -k_1 \frac{\partial u_1^o}{\partial n} - \gamma u_1^o + k_2 \frac{\partial u_2^o}{\partial n} + \gamma u_2^o \quad \in \mathcal{H}^{-1/2}(\Gamma)$$

The expression of the Steklov-Poincaré operator of the problem is: Find $\phi \in \mathcal{H}^{1/2}(\Gamma)$ such that

$$\mathcal{S}\phi = \mathcal{G}$$

The discrete version of the Steklov-Poincaré operator is the Schur complement which is obtained by considering the matrix form of the problem given by equation (34), where the first equation gives:

$$\boldsymbol{U}_1 = \boldsymbol{A}_{11}^{-1}(\boldsymbol{F}_1 - \boldsymbol{A}_{1\Gamma}\boldsymbol{U}_\Gamma) \tag{36}$$

the third equation gives:

$$\boldsymbol{U}_2 = \boldsymbol{A}_{22}^{-1}(\boldsymbol{F}_2 - \boldsymbol{A}_{2\Gamma}\boldsymbol{U}_\Gamma) \tag{37}$$

and the second equations gives:

$$\boldsymbol{A}_{\Gamma 1}\boldsymbol{U}_1 + \boldsymbol{A}_{\Gamma\Gamma}\boldsymbol{U}_\Gamma + \boldsymbol{A}_{\Gamma 2}\boldsymbol{U}_2 = \boldsymbol{F}_\Gamma \tag{38}$$

Eventually, after substituting (36) and (37) into (38), the following equation is obtained:

$$\left(\boldsymbol{A}_{\Gamma\Gamma} - \boldsymbol{A}_{\Gamma 1}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{1\Gamma} - \boldsymbol{A}_{\Gamma 2}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{2\Gamma}\right)\boldsymbol{U}_{\Gamma} = \boldsymbol{F}_{\Gamma} - \boldsymbol{A}_{\Gamma 1}\boldsymbol{A}_{11}^{-1}\boldsymbol{F}_{1} - \boldsymbol{A}_{\Gamma 2}\boldsymbol{A}_{22}^{-1}\boldsymbol{F}_{2} \tag{39}$$

which is written as:

$$\boldsymbol{S}\boldsymbol{U}_{\Gamma} = \boldsymbol{G}$$

where $\boldsymbol{S}$ is the Schur complement and is given as:

$$\boxed{\boldsymbol{S} = \boldsymbol{A}_{\Gamma\Gamma} - \boldsymbol{A}_{\Gamma 1}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{1\Gamma} - \boldsymbol{A}_{\Gamma 2}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{2\Gamma}}$$

and

$$\boldsymbol{G} = \boldsymbol{F}_{\Gamma} - \boldsymbol{A}_{\Gamma 1}\boldsymbol{A}_{11}^{-1}\boldsymbol{F}_{1} - \boldsymbol{A}_{\Gamma 2}\boldsymbol{A}_{22}^{-1}\boldsymbol{F}_{2}$$

(d) To identify the preconditioner for the Schur complement equation arising from the Dirchlet-Robin iteration-by-subdomain scheme, we first define the following:

$$\boldsymbol{S} = \boldsymbol{S}_{1} + \boldsymbol{S}_{2}$$

where

$$\boldsymbol{S}_{1} = \boldsymbol{A}_{\Gamma\Gamma}^{(1)} - \boldsymbol{A}_{\Gamma 1}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{1\Gamma}$$

$$\boldsymbol{S}_{2} = \boldsymbol{A}_{\Gamma\Gamma}^{(2)} - \boldsymbol{A}_{\Gamma 2}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{2\Gamma}$$

Considering a Gauss-Seidel-type iteration-by-subdomain given by (35), we obtain the following equations:

$$\boldsymbol{U}_{1}^{(k)} = \boldsymbol{A}_{11}^{-1}(\boldsymbol{F}_{1} - \boldsymbol{A}_{1\Gamma}\boldsymbol{U}_{\Gamma}^{(k)}) \tag{40}$$

$$\boldsymbol{U}_{2}^{(k-1)} = \boldsymbol{A}_{22}^{-1}(\boldsymbol{F}_{2} - \boldsymbol{A}_{2\Gamma}\boldsymbol{U}_{\Gamma}^{(k-1)}) \tag{41}$$

$$\boldsymbol{A}_{\Gamma 1}\boldsymbol{U}_{1}^{(k)} + \boldsymbol{A}_{\Gamma\Gamma}^{(1)}\boldsymbol{U}_{\Gamma} = \boldsymbol{F}_{\Gamma} - \boldsymbol{A}_{\Gamma 2}\boldsymbol{U}_{2}^{(k-1)} - \boldsymbol{A}_{\Gamma\Gamma}^{(2)}\boldsymbol{U}_{\Gamma}^{(k-1)} \tag{42}$$

Substituting (40) and (41) into (42) yields:

$$\boldsymbol{A}_{\Gamma 1}\boldsymbol{A}_{11}^{-1}(\boldsymbol{F}_{1} - \boldsymbol{A}_{1\Gamma}\boldsymbol{U}_{\Gamma}^{(k)}) + \boldsymbol{A}_{\Gamma\Gamma}^{(1)}\boldsymbol{U}_{\Gamma}^{(k)} = \boldsymbol{F}_{\Gamma} - \boldsymbol{A}_{\Gamma 2}\boldsymbol{A}_{22}^{-1}(\boldsymbol{F}_{2} - \boldsymbol{A}_{2\Gamma}\boldsymbol{U}_{\Gamma}^{(k-1)}) - \boldsymbol{A}_{\Gamma\Gamma}^{(2)}\boldsymbol{U}_{\Gamma}^{(k-1)} \tag{43}$$

Re-arranging yields:

$$(\boldsymbol{A}_{\Gamma\Gamma}^{(1)} - \boldsymbol{A}_{\Gamma 1}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{1\Gamma})\boldsymbol{U}_{\Gamma}^{(k)} = \boldsymbol{F}_{\Gamma} - \boldsymbol{A}_{\Gamma 1}\boldsymbol{A}_{11}^{-1}\boldsymbol{F}_{1} - \boldsymbol{A}_{\Gamma 2}\boldsymbol{A}_{22}^{-1}\boldsymbol{F}_{2} - (\boldsymbol{A}_{\Gamma\Gamma}^{(2)} - \boldsymbol{A}_{\Gamma 2}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{2\Gamma})\boldsymbol{U}_{\Gamma}^{(k-1)} \tag{44}$$

That is equivalent to:

$$\begin{aligned}
\boldsymbol{S}_{1}\boldsymbol{U}_{\Gamma}^{(k)} &= \boldsymbol{G} - \boldsymbol{S}_{2}\boldsymbol{U}_{\Gamma}^{(k-1)} \\
&= \boldsymbol{G} - \boldsymbol{S}\boldsymbol{U}_{\Gamma}^{(k-1)} + \boldsymbol{S}_{1}\boldsymbol{U}_{\Gamma}^{(k-1)}
\end{aligned} \tag{45}$$

Inverting the matrix $\boldsymbol{S}_{1}$ yields:

$$\boldsymbol{U}_{\Gamma}^{(k)} = \boldsymbol{U}_{\Gamma}^{(k-1)} + \boldsymbol{S}_{1}^{-1}(\boldsymbol{G} - \boldsymbol{S}\boldsymbol{U}_{\Gamma}^{(k-1)}) \tag{46}$$

which is a Richardson iteration for the Schur complement equation with preconditioner $\boldsymbol{P} = \boldsymbol{S}_{1}$

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3 Coupling in space of heterogeneous problems

3.1 Question 1: Elasticity and beam-theory

(a) Taking into account the following definition:

$$\boldsymbol{\nabla}_S = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$$

Considering also the definitions of the displacement vector

$$\boldsymbol{u} = [u, v]^T,$$

the strain vector

$$\boldsymbol{\epsilon} = [\epsilon_{xx}, \epsilon_{yy}, \gamma_{xy}]^T = [\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}]^T = \boldsymbol{\nabla}_S \boldsymbol{u},$$

the stress tensor

$$\boldsymbol{\tau} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix},$$

and the stress vector

$$\boldsymbol{\sigma} = [\sigma_{xx}, \sigma_{yy}, \tau_{xy}]^T$$

In addition, the constitutive law for the plane stress is $\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$ where the matrix $\boldsymbol{D}$ is given by:

$$\boldsymbol{D} = \dfrac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu)/2 \end{bmatrix}$$

where E and ν are the Young's modulus of elasticity and Poisson's ration, respectively.

The equilibrium equations or the momentum equations for elasticity are given by:

$$\boxed{\boldsymbol{\nabla}_S^T \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}}$$

where $\boldsymbol{f} = [f_x, f_y]^T$ is the body force vector.

For the problem of interest where the square wall $[0, L] \times [-L, 0]$ is clamped at all its sides except the top side, the boundary conditions are:

Homogeneous Dirchlet on the bottom, left and right sides, i.e.

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } (x, -L), (0, y), (L, y)$$

and Neumann on the top side, that is the normal traction coming from the beam:

$$\boldsymbol{\tau}\boldsymbol{n} = \boldsymbol{T} \text{ on } (x, 0)$$

where $\boldsymbol{n}$ is the outward unit normal to the top side and $\boldsymbol{T} = [T_1, T_2]^T$ is the normal traction on the top side due to the beam load.

(b) Before adding the elastic wall as a support to the beam, the strong form of the beam field equation was:

$$EI\frac{d^4v}{dx^4} = f$$

By adding the elastic wall underneath the beam, the beam field equation is modified to:

$$\boxed{EI\frac{d^4v}{dx^4} = f - R(v)}$$

where the supporting wall is represented by a force $R(v)$ opposite to the beam distributed load f (in $y-$direction).

The boundary conditions of the beam are $v(0) = \dfrac{dv}{dx}(0) = v(L) = \dfrac{dv}{dx}(L) = 0$

(c) At the interface between the beam and the wall (at $y = 0$), the transmission conditions are:

Continuity of the vertical displacements, i.e.

$$\boxed{v_b(x) = v_w(x,0)} \qquad \longrightarrow \qquad \text{strong continuity}$$

Continuity of the normal traction, i.e.

$$[0, R_b]^T = (\boldsymbol{\tau n})_w \qquad \longrightarrow \qquad \text{weak continuity}$$

and since $\boldsymbol{n} = [0, 1]^T$ on the top side of the wall (at $y = 0$), this yields the following transmission condition for the normal traction:

$$\boxed{R_b = (\sigma_{yy})_w}$$

and it is assumed that $\tau_{xy} = 0$ so that the normal traction is in the vertical direction.

(d) For the Euler-Bernoulli beam theory to be valid, it must be satisfied that the horizontal displacement is zero, i.e.

$$\boxed{u_b(x) = u_w(x,0) = 0}$$

and the tangential component of the traction on the wall at $y = 0$ should be also zero, i.e.

$$\boxed{0 = (\tau_{xy})_w}$$

If the horizontal components of the displacement and the traction were not assumed to be zero, Euler-Bernoulli beam theory would not be valid and another theory for the beam would be required.
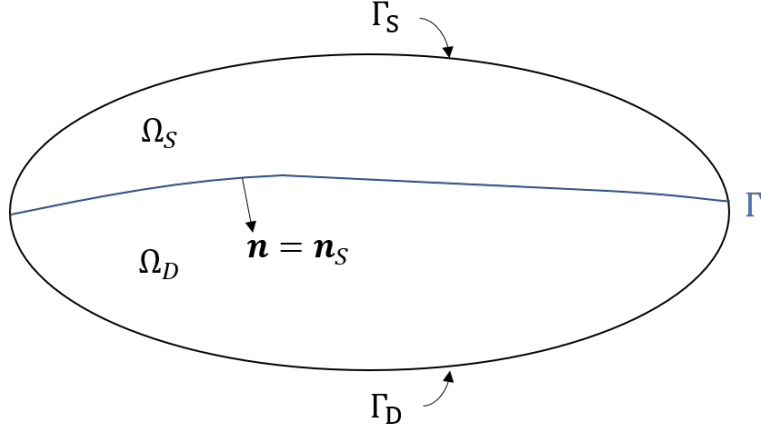
3.2 Question 2: Stoke's and Darcy problems

Figure 3: Stoke's and Darcy's domains with boundaries and interface identified

(a) Stoke's and Darcy's problems are written in their respective subdomains Ω_S and Ω_D, respectively.

$$\begin{aligned} -\nu\boldsymbol{\Delta}\boldsymbol{u}_S + \boldsymbol{\nabla}p_S &= \boldsymbol{f} && \text{in } \Omega_S \\ \boldsymbol{\nabla}\cdot\boldsymbol{u}_S &= 0 && \text{in } \Omega_S \\ \boldsymbol{u}_S &= \overline{\boldsymbol{u}}_S && \text{on } \Gamma_S \end{aligned}$$

$$\begin{aligned} k^{-1}\boldsymbol{u}_D + \boldsymbol{\nabla}\phi &= \boldsymbol{0} && \text{in } \Omega_D \\ \boldsymbol{\nabla}\cdot\boldsymbol{u}_D &= 0 && \text{in } \Omega_D \\ \boldsymbol{n}\cdot\boldsymbol{u}_D &= \overline{u}_{n,D} && \text{on } \Gamma_D \end{aligned}$$

where the interface conditions on Γ are:

$$\begin{aligned} \text{"Strongly"} &\longrightarrow & \boldsymbol{n}\cdot\boldsymbol{u}_S &= \boldsymbol{n}\cdot\boldsymbol{u}_D \\ \text{"Weakly"} &\longrightarrow & p_S - (\boldsymbol{n}\cdot\nu\boldsymbol{\nabla}\boldsymbol{u}_S)\cdot\boldsymbol{n} &= \phi \\ \text{"Weakly"} &\longrightarrow & \boldsymbol{u}_S\cdot\boldsymbol{t} &= -\frac{\sqrt{k}}{\alpha_{BJ}}(\boldsymbol{n}\cdot\nu\boldsymbol{\nabla}\boldsymbol{u}_S)\cdot\boldsymbol{t} \end{aligned}$$

where $\boldsymbol{t}$ is a unit tangential vector on Γ.

First, the weak forms of the Stoke's problem are derived by multiplying the 1st equation (the momentum equation) by an arbitrary vector test function $\delta\boldsymbol{u}_S \in [\mathcal{H}^1(\Omega_S)]^d$ and the 2nd equation (the mass conservation equation) by an arbitrary scalar test function $q_S \in \mathcal{L}_2(\Omega_S)$

$$-\int_{\Omega_S}\delta\boldsymbol{u}_S\cdot\nu\boldsymbol{\Delta}\boldsymbol{u}_S + \int_{\Omega_S}\delta\boldsymbol{u}_S\cdot\boldsymbol{\nabla}p_S = \int_{\Omega_S}\delta\boldsymbol{u}_S\cdot\boldsymbol{f} \tag{47a}$$

$$\int_{\Omega_S}q_s(\boldsymbol{\nabla}\cdot\boldsymbol{u}_S) = 0 \tag{47b}$$

Integrating by parts and setting $\delta\boldsymbol{u}_S = \boldsymbol{0}$ on Γ_S yields

$$\int_{\Omega_S}\boldsymbol{\nabla}\delta\boldsymbol{u}_S : \nu\boldsymbol{\nabla}\boldsymbol{u}_S - \int_{\Omega_S}p_S(\boldsymbol{\nabla}\cdot\delta\boldsymbol{u}_S) - \int_{\Gamma}\delta\boldsymbol{u}_S\cdot\left[\boldsymbol{n}_S\cdot(-p_S\boldsymbol{I}+\nu\boldsymbol{\nabla}\boldsymbol{u}_S)\right] = \int_{\Omega_S}\delta\boldsymbol{u}_S\cdot\boldsymbol{f} \tag{48a}$$

$$\int_{\Omega_S}q_s(\boldsymbol{\nabla}\cdot\boldsymbol{u}_S) = 0 \tag{48b}$$

Second, the weak forms of the Darcy's problem are derived by multiplying the 1st equation (the momentum equation) by an arbitrary vector test function $\delta\boldsymbol{u}_D \in \mathcal{H}^{div}(\Omega_D)$ and the 2nd equation (the mass conservation equation) by an arbitrary scalar test function $q_D \in \mathcal{L}_2(\Omega_D)$

$$\int_{\Omega_D} \delta\boldsymbol{u}_D \cdot k^{-1}\boldsymbol{u}_D + \int_{\Omega_D} \delta\boldsymbol{u}_D \cdot \boldsymbol{\nabla}\phi = 0 \tag{49a}$$

$$\int_{\Omega_D} q_D(\boldsymbol{\nabla} \cdot \boldsymbol{u}_D) = 0 \tag{49b}$$

Integrating by parts and setting $\delta\boldsymbol{u}_D = \boldsymbol{0}$ on Γ_D yields

$$\int_{\Omega_D} \delta\boldsymbol{u}_D \cdot k^{-1}\boldsymbol{u}_D - \int_{\Omega_D} \phi(\boldsymbol{\nabla} \cdot \delta\boldsymbol{u}_D) + \int_{\Gamma} \delta\boldsymbol{u}_D \cdot \phi\boldsymbol{n}_D = 0 \tag{50a}$$

$$\int_{\Omega_D} q_D(\boldsymbol{\nabla} \cdot \boldsymbol{u}_D) = 0 \tag{50b}$$

Employing the following Galerkin finite element approximations for Stoke's problem:

$$\boldsymbol{u}_S \approx \boldsymbol{u}_S^h = \sum_{i=1}^{n_{sd}} u_{S,i}^h \boldsymbol{e}_i = \sum_{i=1}^{n_{sd}} \sum_A N_A\, U_{S,i,A}\, \boldsymbol{e}_i, \tag{51a}$$

$$\delta\boldsymbol{u}_S \approx \delta\boldsymbol{u}_S^h = \sum_{i=1}^{n_{sd}} \delta u_{S,i}^h \boldsymbol{e}_i = \sum_{i=1}^{n_{sd}} \sum_B N_B\, \delta U_{S,i,B}\, \boldsymbol{e}_i, \tag{51b}$$

$$p_S \approx p_S^h = \sum_{\widehat{A}} \widehat{N}_{\widehat{A}}\, P_{\widehat{A}}, \tag{51c}$$

$$q_S \approx q_S^h = \sum_{\widehat{A}} \widehat{N}_{\widehat{A}}\, Q_{S,\widehat{A}}. \tag{51d}$$

and the following approximations for Darcy's problem:

$$\boldsymbol{u}_D \approx \boldsymbol{u}_D^h = \sum_{i=1}^{n_{sd}} u_{D,i}^h \boldsymbol{e}_i = \sum_{i=1}^{n_{sd}} \sum_A N_A\, U_{D,i,A}\, \boldsymbol{e}_i, \tag{52a}$$

$$\delta\boldsymbol{u}_D \approx \delta\boldsymbol{u}_D^h = \sum_{i=1}^{n_{sd}} \delta u_{D,i}^h \boldsymbol{e}_i = \sum_{i=1}^{n_{sd}} \sum_B N_B\, \delta U_{D,i,B}\, \boldsymbol{e}_i, \tag{52b}$$

$$\phi \approx \phi^h = \sum_{\widehat{A}} \widehat{N}_{\widehat{A}}\, \Phi_{\widehat{A}}, \tag{52c}$$

$$q_D \approx q_D^h = \sum_{\widehat{A}} \widehat{N}_{\widehat{A}}\, Q_{D,\widehat{A}}. \tag{52d}$$

This will give rise to the matrix form of the Stokes's problem as [2]:

$$\begin{bmatrix} \boldsymbol{K}_S & \boldsymbol{G}_S \\ \boldsymbol{G}_S^T & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U}_S \\ \boldsymbol{P}_S \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f}_S \\ \boldsymbol{h}_S \end{Bmatrix}$$

and the matrix form of the Darcy's problem as:

$$\begin{bmatrix} \boldsymbol{M}_D & \boldsymbol{G}_D \\ \boldsymbol{G}_D^T & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U}_D \\ \boldsymbol{\Phi}_D \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f}_D \\ \boldsymbol{h}_D \end{Bmatrix}$$

By separating the degrees of freedom of the velocity into interior and interface, i.e $\boldsymbol{U} = [\boldsymbol{U}^{int^T}, \boldsymbol{\lambda}^T]^T$, the matrix forms of the Stoke's (left) and Darcy's (right) problems are written as:

$$\begin{bmatrix} \boldsymbol{A}_{SS} & \boldsymbol{A}_{S\Gamma} & \boldsymbol{B}_{SS} \\ \boldsymbol{A}_{\Gamma S} & \boldsymbol{A}_{\Gamma\Gamma}^{(S)} & \boldsymbol{B}_{S\Gamma} \\ \boldsymbol{B}_{SS}^T & \boldsymbol{B}_{\Gamma S} & \boldsymbol{0} \end{bmatrix} \left\{ \begin{array}{c} \boldsymbol{U}_S^{int} \\ \boldsymbol{\lambda} \\ \boldsymbol{P}_S \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{f}_{SS} \\ \boldsymbol{f}_{S\Gamma} \\ \boldsymbol{h}_S \end{array} \right\} \qquad \begin{bmatrix} \boldsymbol{A}_{DD} & \boldsymbol{A}_{D\Gamma} & \boldsymbol{B}_{DD} \\ \boldsymbol{A}_{\Gamma D} & \boldsymbol{A}_{\Gamma\Gamma}^{(D)} & \boldsymbol{B}_{D\Gamma} \\ \boldsymbol{B}_{DD}^T & \boldsymbol{B}_{\Gamma D} & \boldsymbol{0} \end{bmatrix} \left\{ \begin{array}{c} \boldsymbol{U}_D^{int} \\ \boldsymbol{\lambda} \\ \boldsymbol{\Phi}_D \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{f}_{DD} \\ \boldsymbol{f}_{D\Gamma} \\ \boldsymbol{h}_D \end{array} \right\}$$

By combining the two problems together in one matrix form, i.e. monolithic scheme, it yields:

$$\begin{bmatrix} \boldsymbol{A}_{SS} & \boldsymbol{B}_{SS} & \boldsymbol{A}_{S\Gamma} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{B}_{SS}^T & \boldsymbol{0} & \boldsymbol{B}_{\Gamma S} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{A}_{\Gamma S} & \boldsymbol{B}_{S\Gamma} & \boldsymbol{A}_{\Gamma\Gamma}^{(S)} + \boldsymbol{A}_{\Gamma\Gamma}^{(D)} & \boldsymbol{A}_{\Gamma D} & \boldsymbol{B}_{D\Gamma} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_{D\Gamma} & \boldsymbol{A}_{DD} & \boldsymbol{B}_{DD} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{B}_{\Gamma D} & \boldsymbol{B}_{DD}^T & \boldsymbol{0} \end{bmatrix} \left\{ \begin{array}{c} \boldsymbol{U}_S^{int} \\ \boldsymbol{P}_S \\ \boldsymbol{\lambda} \\ \boldsymbol{U}_D^{int} \\ \boldsymbol{\Phi}_D \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{f}_{SS} \\ \boldsymbol{h}_S \\ \boldsymbol{f}_{S\Gamma} + \boldsymbol{f}_{D\Gamma} \\ \boldsymbol{f}_{DD} \\ \boldsymbol{h}_D \end{array} \right\} \qquad (53)$$

By combining all the degrees of freedom of velocity and pressure in each subdomain as $\boldsymbol{\mathcal{U}}_S = [\boldsymbol{U}_S^{int^T}, \boldsymbol{P}_S^T]^T$ and $\boldsymbol{\mathcal{U}}_D = [\boldsymbol{U}_D^{int^T}, \boldsymbol{\Phi}_D^T]^T$, the matrix form is further simplified to:

$$\begin{bmatrix} \boldsymbol{\mathcal{A}}_{SS} & \boldsymbol{\mathcal{A}}_{S\Gamma} & \boldsymbol{0} \\ \boldsymbol{\mathcal{A}}_{\Gamma S} & \boldsymbol{\mathcal{A}}_{\Gamma\Gamma} & \boldsymbol{\mathcal{A}}_{\Gamma D} \\ \boldsymbol{0} & \boldsymbol{\mathcal{A}}_{D\Gamma} & \boldsymbol{\mathcal{A}}_{DD} \end{bmatrix} \left\{ \begin{array}{c} \boldsymbol{\mathcal{U}}_S \\ \boldsymbol{\lambda} \\ \boldsymbol{\mathcal{U}}_D \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{F}_S \\ \boldsymbol{F}_\Gamma \\ \boldsymbol{F}_D \end{array} \right\}$$

the first equation gives:

$$\boldsymbol{\mathcal{U}}_S = \boldsymbol{\mathcal{A}}_{SS}^{-1}(\boldsymbol{F}_S - \boldsymbol{\mathcal{A}}_{S\Gamma}\boldsymbol{\lambda}) \qquad (54)$$

the third equation gives:

$$\boldsymbol{\mathcal{U}}_D = \boldsymbol{\mathcal{A}}_{DD}^{-1}(\boldsymbol{F}_D - \boldsymbol{\mathcal{A}}_{D\Gamma}\boldsymbol{\lambda}) \qquad (55)$$

and the second equations gives:

$$\boldsymbol{\mathcal{A}}_{\Gamma S}\boldsymbol{\mathcal{U}}_S + \boldsymbol{\mathcal{A}}_{\Gamma\Gamma}\boldsymbol{\lambda} + \boldsymbol{\mathcal{A}}_{\Gamma D}\boldsymbol{\mathcal{U}}_D = \boldsymbol{F}_\Gamma \qquad (56)$$

Eventually, after substituting (54) and (55) into (56), the following equation is obtained:

$$(\boldsymbol{\mathcal{A}}_{\Gamma\Gamma} - \boldsymbol{\mathcal{A}}_{\Gamma S}\boldsymbol{\mathcal{A}}_{SS}^{-1}\boldsymbol{\mathcal{A}}_{S\Gamma} - \boldsymbol{\mathcal{A}}_{\Gamma D}\boldsymbol{\mathcal{A}}_{DD}^{-1}\boldsymbol{\mathcal{A}}_{D\Gamma})\,\boldsymbol{\lambda} = \boldsymbol{F}_\Gamma - \boldsymbol{\mathcal{A}}_{\Gamma S}\boldsymbol{\mathcal{A}}_{SS}^{-1}\boldsymbol{F}_1 - \boldsymbol{\mathcal{A}}_{\Gamma D}\boldsymbol{\mathcal{A}}_{DD}^{-1}\boldsymbol{F}_2 \qquad (57)$$

which is written as:

$$(\boldsymbol{\mathcal{S}}_S - \boldsymbol{\mathcal{S}}_D)\boldsymbol{\lambda} = \boldsymbol{G}$$

where

$$\boldsymbol{\mathcal{S}}_S = \boldsymbol{A}_{\Gamma\Gamma}^{(S)} - \boldsymbol{\mathcal{A}}_{\Gamma S}\boldsymbol{\mathcal{A}}_{SS}^{-1}\boldsymbol{\mathcal{A}}_{S\Gamma}$$

$$\boldsymbol{\mathcal{S}}_D = \boldsymbol{\mathcal{A}}_{\Gamma D}\boldsymbol{\mathcal{A}}_{DD}^{-1}\boldsymbol{\mathcal{A}}_{D\Gamma} - \boldsymbol{A}_{\Gamma\Gamma}^{(D)}$$

$$\boldsymbol{G} = \boldsymbol{F}_\Gamma - \boldsymbol{\mathcal{A}}_{\Gamma S}\boldsymbol{\mathcal{A}}_{SS}^{-1}\boldsymbol{F}_1 - \boldsymbol{\mathcal{A}}_{\Gamma D}\boldsymbol{\mathcal{A}}_{DD}^{-1}\boldsymbol{F}_2$$

and for the problem at hand, $\boldsymbol{G}$ is given to be $\boldsymbol{0}$ in the question.

(b) The Dirchlet-Neumann iteration-by-subdomain scheme is written as:

$$
\begin{aligned}
-\nu \boldsymbol{\Delta} \boldsymbol{u}_S^{(k)} + \boldsymbol{\nabla} p_S^{(k)} &= \boldsymbol{f} && \text{in } \Omega_S \\
\boldsymbol{\nabla} \cdot \boldsymbol{u}_S^{(k)} &= 0 && \text{in } \Omega_S \\
\boldsymbol{u}_S^{(k)} &= \overline{\boldsymbol{u}}_S && \text{on } \Gamma_S \\
\boldsymbol{n} \cdot \boldsymbol{u}_S^{(k)} &= \boldsymbol{n} \cdot \boldsymbol{u}_D^{(k-1)} && \text{on } \Gamma \\
\boldsymbol{u}_S^{(k)} \cdot \boldsymbol{t} &= -\frac{\sqrt{k}}{\alpha_{BJ}} (\boldsymbol{n} \cdot \nu \boldsymbol{\nabla} \boldsymbol{u}_S^{(k)}) \cdot \boldsymbol{t} && \text{on } \Gamma
\end{aligned}
$$

$$
\begin{aligned}
k^{-1} \boldsymbol{u}_D^{(k)} + \boldsymbol{\nabla} \phi^{(k)} &= \boldsymbol{0} && \text{in } \Omega_D \\
\boldsymbol{\nabla} \cdot \boldsymbol{u}_D^{(k)} &= 0 && \text{in } \Omega_D \\
\boldsymbol{n} \cdot \boldsymbol{u}_D^{(k)} &= \overline{u}_{n,D} && \text{on } \Gamma_D \\
\phi^{(k)} &= p_S^{(l)} - (\boldsymbol{n} \cdot \nu \boldsymbol{\nabla} \boldsymbol{u}_S^{(l)}) \cdot \boldsymbol{n} && \text{on } \Gamma
\end{aligned}
$$

Again, $l = k - 1$ yields a Jacobi scheme which allows for parallel solve, while $l = k$ yields a Gauss-Seidel scheme with sequential solve.

Looking into equation (53), the matrix form of the scheme is:

$$
\begin{bmatrix} \boldsymbol{A}_{SS} & \boldsymbol{B}_{SS} & \boldsymbol{A}_{S\Gamma} \\ \boldsymbol{B}_{SS}^T & \boldsymbol{0} & \boldsymbol{B}_{\Gamma S} \\ \boldsymbol{A}_{\Gamma S} & \boldsymbol{B}_{S\Gamma} & \boldsymbol{A}_{\Gamma\Gamma}^{(S)} \end{bmatrix} \left\{ \begin{array}{c} \boldsymbol{U}_S^{int^{(k)}} \\ \boldsymbol{P}_S^{(k)} \\ \boldsymbol{\lambda}^{(k)} \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{f}_{SS} \\ \boldsymbol{h}_S \\ \boldsymbol{F}_\Gamma - \boldsymbol{A}_{\Gamma\Gamma}^{(D)} \boldsymbol{\lambda}^{(k-1)} - \boldsymbol{A}_{\Gamma D} \boldsymbol{U}_D^{int^{(k-1)}} \end{array} \right\}
$$

$$
\begin{bmatrix} \boldsymbol{A}_{DD} & \boldsymbol{B}_{DD} \\ \boldsymbol{B}_{DD}^T & \boldsymbol{0} \end{bmatrix} \left\{ \begin{array}{c} \boldsymbol{U}_D^{int^{(k)}} \\ \boldsymbol{\Phi}_D^{(k)} \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{f}_{DD} - \boldsymbol{A}_{D\Gamma} \boldsymbol{\lambda}^{(l)} \\ \boldsymbol{h}_D \end{array} \right\}
$$

(c) The Richardson iteration scheme takes the form:

$$
\left\{ \begin{array}{c} \boldsymbol{U}_S^{int^{(k)}} \\ \boldsymbol{P}_S^{(k)} \\ \boldsymbol{\lambda}^{(k)} \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{U}_S^{int^{(k-1)}} \\ \boldsymbol{P}_S^{(k-1)} \\ \boldsymbol{\lambda}^{(k-1)} \end{array} \right\} + \left(\boldsymbol{G}_S - \begin{bmatrix} \boldsymbol{A}_{SS} & \boldsymbol{B}_{SS} & \boldsymbol{A}_{S\Gamma} \\ \boldsymbol{B}_{SS}^T & \boldsymbol{0} & \boldsymbol{B}_{\Gamma S} \\ \boldsymbol{A}_{\Gamma S} & \boldsymbol{B}_{S\Gamma} & \boldsymbol{A}_{\Gamma\Gamma}^{(S)} \end{bmatrix} \left\{ \begin{array}{c} \boldsymbol{U}_S^{int^{(k-1)}} \\ \boldsymbol{P}_S^{(k-1)} \\ \boldsymbol{\lambda}^{(k-1)} \end{array} \right\} \right)
$$

$$
\left\{ \begin{array}{c} \boldsymbol{U}_D^{int^{(k)}} \\ \boldsymbol{\Phi}_D^{(k)} \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{U}_D^{int^{(k-1)}} \\ \boldsymbol{\Phi}_D^{(k-1)} \end{array} \right\} + \left(\left\{ \begin{array}{c} \boldsymbol{f}_{DD} - \boldsymbol{A}_{D\Gamma} \boldsymbol{\lambda}^{(l)} \\ \boldsymbol{h}_D \end{array} \right\} - \begin{bmatrix} \boldsymbol{A}_{DD} & \boldsymbol{B}_{DD} \\ \boldsymbol{B}_{DD}^T & \boldsymbol{0} \end{bmatrix} \left\{ \begin{array}{c} \boldsymbol{U}_D^{int^{(k-1)}} \\ \boldsymbol{\Phi}_D^{(k-1)} \end{array} \right\} \right)
$$

where

$$
\boldsymbol{G}_S = \left\{ \begin{array}{c} \boldsymbol{f}_{SS} \\ \boldsymbol{h}_S \\ \boldsymbol{F}_\Gamma - \boldsymbol{A}_{\Gamma\Gamma}^{(D)} \boldsymbol{\lambda}^{(k-1)} - \boldsymbol{A}_{\Gamma D} \boldsymbol{U}_D^{int^{(k-1)}} \end{array} \right\}
$$

4 Coupling in time: Monolithic and partitioned schemes

4.1 Question 1: Discretization of transient heat transfer equation

Considering the one-dimensional, transient, heat transfer equation:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f \quad \text{in } [0,1]$$
$$u(x = 0, t) = 0$$
$$u(x = 1, t) = 0$$
$$u(x, t = 0) = 0$$

The weak form is obtained after pre-multiplying by a test function $v \in \mathcal{H}^1(\Omega)$ and integrating over the whole domain Ω:

$$\left(v, \frac{\partial u}{\partial t}\right)_\Omega - \left(v, \kappa \frac{\partial^2 u}{\partial x^2}\right)_\Omega = \left(v, f\right)_\Omega$$

Integrating by parts the 2nd term on the LHS yields:

$$\left(v, \frac{\partial u}{\partial t}\right)_\Omega + \left(\frac{\partial v}{\partial x}, \kappa \frac{\partial u}{\partial x}\right)_\Omega - \left\langle v, \kappa \frac{\partial u}{\partial x} n \right\rangle_{\partial\Omega} = \left(v, f\right)_\Omega$$

Since we only have Dirchlet boundary conditions on $\partial\Omega$, therefore, the boundary term is eliminated, which yields the final weak form as:

$$\left(v, \frac{\partial u}{\partial t}\right)_\Omega + \left(\frac{\partial v}{\partial x}, \kappa \frac{\partial u}{\partial x}\right)_\Omega = \left(v, f\right)_\Omega$$

Using Galerkin finite elements, the solution u and the test function v are approximated as:

$$u(x, t) \approx u_h(x, t) = \sum_{i=1} U_i(t) N_i(x) = \boldsymbol{N}\boldsymbol{U}$$

$$v(x, t) \approx v_h(x, t) = \sum_{i=1} V_i(t) N_i(x) = \boldsymbol{V}^T \boldsymbol{N}^T$$

where $\boldsymbol{U}$ is a column-vector containing the nodal values of u, $\boldsymbol{V}$ is a column-vector containing the nodal values of v, and $\boldsymbol{N}$ is a row-vector of the nodal shape functions. By using the FE approximations into the weak form we obtain:

$$\int_0^1 \boldsymbol{N}^T \boldsymbol{N} \, d\Omega \, \frac{d\boldsymbol{U}}{dt} + \int_0^1 \kappa \frac{\partial \boldsymbol{N}^T}{\partial x} \frac{\partial \boldsymbol{N}}{\partial x} \, d\Omega \, \boldsymbol{U} = \int_0^1 \boldsymbol{N}^T f \, d\Omega$$

which results in the following algebric problem:

$$\boldsymbol{M} \frac{d\boldsymbol{U}}{dt} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F}$$

where $\boldsymbol{M}$ is called mass matrix, $\boldsymbol{K}$ is the stiffness matrix and $\boldsymbol{F}$ is the right hand side forcing vector.

The time discretization using BDF1 yields the following system of equations:

$$\boldsymbol{M} \frac{\boldsymbol{U}^{n+1} - \boldsymbol{U}^n}{\delta t} + \boldsymbol{K}\boldsymbol{U}^{n+1} = \boldsymbol{F}^{n+1}$$

Considering a time step of $\delta t = 1$ and a source term $f = 1$ (not time-dependent), the system of equations is written as:

$$(\boldsymbol{M} + \boldsymbol{K})\boldsymbol{U}^{n+1} = \boldsymbol{F} + \boldsymbol{M}\boldsymbol{U}^n$$

Let $\boldsymbol{A} = \boldsymbol{M} + \boldsymbol{K}$ and $\boldsymbol{B} = \boldsymbol{F} + \boldsymbol{M}\boldsymbol{U}^n$, the nodal solution at time step $n + 1$ is obtained as:

$$\boldsymbol{U}^{n+1} = \boldsymbol{A}^{-1}\boldsymbol{B} \tag{58}$$

4.2 Question 2: Domain decomposition approach using Monolithic scheme

By splitting the domain into two sub-domains $\Omega_1 = [0, 0.4]$ and $\Omega_2 = [0.4, 1]$, we define an interface Γ at $x = 0.4$ (corresponding to the nodal value u_2). Furthermore, the weak form of the problem is written for the two subdomains as follows:

Sub-domain 1:

$$\left(v, \frac{\partial u}{\partial t}\right)_{\Omega_1} + \left(\frac{\partial v}{\partial x}, \kappa\frac{\partial u}{\partial x}\right)_{\Omega_1} - \left\langle v, \kappa\frac{\partial u}{\partial x}n_1\right\rangle_\Gamma = (v, f)_{\Omega_1}$$

Sub-domain 2:

$$\left(v, \frac{\partial u}{\partial t}\right)_{\Omega_2} + \left(\frac{\partial v}{\partial x}, \kappa\frac{\partial u}{\partial x}\right)_{\Omega_2} - \left\langle v, \kappa\frac{\partial u}{\partial x}n_2\right\rangle_\Gamma = (v, f)_{\Omega_2}$$

with the transmission conditions:

$$\boxed{[\![u]\!]_\Gamma = 0} \longrightarrow \text{1st transmission condition - strong continuity}$$

$$\boxed{[\![\kappa\frac{\partial u}{\partial x}n]\!]_\Gamma = 0} \longrightarrow \text{2nd transmission condition - weak continuity}$$

In order to solve the problem in a monolithic way, the two equations for the two subdomains are summed which yields:

$$\left(v, \frac{\partial u}{\partial t}\right)_\Omega + \left(\frac{\partial v}{\partial x}, \kappa\frac{\partial u}{\partial x}\right)_\Omega - \underbrace{\left[\left\langle v, \kappa\frac{\partial u}{\partial x}n_1\right\rangle_\Gamma + \left\langle v, \kappa\frac{\partial u}{\partial x}n_2\right\rangle_\Gamma\right]}_{=0} = (v, f)_\Omega$$

where the boundary integrals at the interface Γ are equal to zero due to the 2nd transmission condition. ($n_1 = -n_2$ has been used)

4.3 Question 3: Algebraic form of the Dirchlet-to-Neumann operator for the left sub-domain

Considering the mesh with 6 nodes described in the question, the system of equations given earlier by (58) is written as:

$$
\begin{bmatrix}
A_{00} & A_{01} & 0 & 0 & 0 & 0 \\
A_{10} & A_{11} & A_{12} & 0 & 0 & 0 \\
0 & A_{21} & A_{22} & A_{23} & 0 & 0 \\
0 & 0 & A_{32} & A_{33} & A_{34} & 0 \\
0 & 0 & 0 & A_{43} & A_{44} & A_{45} \\
0 & 0 & 0 & 0 & A_{54} & A_{55}
\end{bmatrix}
\begin{Bmatrix}
U_0^{n+1} \\
U_1^{n+1} \\
U_2^{n+1} \\
U_3^{n+1} \\
U_4^{n+1} \\
U_5^{n+1}
\end{Bmatrix}
=
\begin{Bmatrix}
B_0^n \\
B_1^n \\
B_2^n \\
B_3^n \\
B_4^n \\
B_5^n
\end{Bmatrix}
$$

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The nodal values of U_0 and U_5 are prescribed by Dirchlet boundary conditions and both are equal to zero. Imposing them strongly yields the following reduced system of equations:

$$
\begin{bmatrix}
A_{11} & A_{12} & 0 & 0 \\
A_{21} & A_{22} & A_{23} & 0 \\
0 & A_{32} & A_{33} & A_{34} \\
0 & 0 & A_{43} & A_{44}
\end{bmatrix}
\begin{Bmatrix}
U_1^{n+1} \\
U_2^{n+1} \\
U_3^{n+1} \\
U_4^{n+1}
\end{Bmatrix}
=
\begin{Bmatrix}
B_1^n \\
B_2^n \\
B_3^n \\
B_4^n
\end{Bmatrix}
$$

The system of equations for the left sub-domain (Ω_1) is:

$$
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{Bmatrix}
U_1^{n+1} \\
U_2^{n+1}
\end{Bmatrix}
=
\begin{Bmatrix}
B_1^n \\
B_2^n
\end{Bmatrix}
$$

Departing from given values U_i^n and an interface value U_2^{n+1}, the Dirchlet-to-Neumann operator for the left sub-domain (Ω_1) is:

$$
A_{11}U_1^{n+1} = B_1^n \boxed{-A_{12}U_2^{n+1}} \quad \longleftarrow \quad \text{Dirchlet condition}
$$

4.4 Question 4: Algebraic form of the Neumann-to-Dirchlet operator for the right sub-domain

Departing from given values of u_i^n and an interface value for the fluxes $\phi^{n+1} = \kappa\partial_x u^{n+1}$ at the coordinate of node 2, the system of equations for the right sub-domain (Ω_2) is:

$$
\begin{bmatrix}
A_{22}^{(\Omega_2)} & A_{23} & 0 \\
A_{32} & A_{33} & A_{34} \\
0 & A_{43} & A_{44}
\end{bmatrix}
\begin{Bmatrix}
U_2^{n+1} \\
U_3^{n+1} \\
U_4^{n+1}
\end{Bmatrix}
=
\begin{Bmatrix}
B_2^n \boxed{-A_{21}U_1^{n+1} - A_{22}^{(\Omega_1)}U_2^{n+1}} \\
B_3^n \\
B_4^n
\end{Bmatrix}
\quad \longleftarrow \quad \text{Neumann flux from } \Omega_1
$$

4.5 Question 5: Iterative algorithm for a staggered approach

For a staggered approach, we define a prediction $\widetilde{U}^{n+1}$ to replace all the unknowns on the RHS of the equations. This allows for parallel computing which is faster but less accurate. The prediction is defined as:

$$
\widetilde{U} = U^n \quad \longleftarrow \quad \text{1st order approximation}
$$

$$
\widetilde{U} = 2U^n - U^{n-1} \quad \longleftarrow \quad \text{2nd order approximation}
$$

The iterative scheme is written as follows: For each time step $n+1$, we iterate k until convergence:

Sub-domain 2:

$$
\begin{bmatrix}
A_{22}^{(\Omega_2)} & A_{23} & 0 \\
A_{32} & A_{33} & A_{34} \\
0 & A_{43} & A_{44}
\end{bmatrix}
\begin{Bmatrix}
U_2^{(n+1)(k)} \\
U_3^{(n+1)(k)} \\
U_4^{(n+1)(k)}
\end{Bmatrix}
=
\begin{Bmatrix}
B_2^n - A_{21}\widetilde{U}_1^{(n+1)(k-1)} - A_{22}^{(\Omega_1)}\widetilde{U}_2^{(n+1)(k-1)} \\
B_3^n \\
B_4^n
\end{Bmatrix}
$$

sub-domain 1:

$$
A_{11}U_1^{(n+1)(k)} = B_1^n - A_{12}\widetilde{U}_2^{(n+1)(k)}
$$

If we reach convergence, we recover the solution of the monolithic problem. However, convergence or stability of this scheme is not guaranteed.

4.6 Question 6: A Substitution and an iteration-by-subdomains scheme

Substitution scheme:
The idea is based on only predicting the unknowns in one subdomain. This yields a serial scheme (no parallelization) which is slower than the staggered approach but with higher accuracy.

The iterative scheme is written as follows: For each time step $n + 1$, we iterate k until convergence:

Sub-domain 2:

$$\begin{bmatrix} A_{22}^{(\Omega_2)} & A_{23} & 0 \\ A_{32} & A_{33} & A_{34} \\ 0 & A_{43} & A_{44} \end{bmatrix} \left\{ \begin{array}{c} U_2^{(n+1)(k)} \\ U_3^{(n+1)(k)} \\ U_4^{(n+1)(k)} \end{array} \right\} = \left\{ \begin{array}{c} B_2^n - A_{21}\widetilde{U}_1^{(n+1)(k-1)} - A_{22}^{(\Omega_1)}\widetilde{U}_2^{(n+1)(k-1)} \\ B_3^n \\ B_4^n \end{array} \right\}$$

sub-domain 1:

$$A_{11}U_1^{(n+1)(k)} = B_1^n - A_{12}U_2^{(n+1)(k)}$$

If we reach convergence, we recover the solution of the monolithic problem. However, convergence or stability of this scheme is not guaranteed.

Iteration-by-subdomains scheme:
Here, the problem is solved iteratively without predictions. The iteration-by-subdomains scheme is written as follows: For each time step $n + 1$, the following equations are solved by iterating k until convergence:

Sub-domain 2:

$$\begin{bmatrix} A_{22}^{(\Omega_2)} & A_{23} & 0 \\ A_{32} & A_{33} & A_{34} \\ 0 & A_{43} & A_{44} \end{bmatrix} \left\{ \begin{array}{c} U_2^{(n+1)(k)} \\ U_3^{(n+1)(k)} \\ U_4^{(n+1)(k)} \end{array} \right\} = \left\{ \begin{array}{c} B_2^n - A_{21}U_1^{(n+1)(k-1)} - A_{22}^{(\Omega_1)}U_2^{(n+1)(k-1)} \\ B_3^n \\ B_4^n \end{array} \right\}$$

sub-domain 1:

$$A_{11}U_1^{(n+1)(k)} = B_1^n - A_{12}U_2^{(n+1)(l)}$$

where $l = k - 1$ yields a Jacobi scheme which allows for parallel solve, while $l = k$ yields a Gauss-Seidel scheme with sequential solve.

4.7 Question 7: Imposing Dirchlet boundary condition on the left sub-domain using Nitche's method

The weak form of the problem in sub-domain Ω_1 was derived earlier as:

$$\left(v, \frac{\partial u}{\partial t}\right)_{\Omega_1} + \left(\frac{\partial v}{\partial x}, \kappa\frac{\partial u}{\partial x}\right)_{\Omega_1} - \left\langle v, \kappa\frac{\partial u}{\partial x}n_1\right\rangle_{\partial\Omega_1} = \left(v, f\right)_{\Omega_1}$$

After adding the terms associated to Nitche's method to impose the Dirchlet boundary conditions weakly in a symmetric way, we obtain the following weak form:

$$\left(v, \frac{\partial u}{\partial t}\right)_{\Omega_1} + \left(\frac{\partial v}{\partial x}, \kappa\frac{\partial u}{\partial x}\right)_{\Omega_1} - \left\langle v, \kappa\frac{\partial u}{\partial x}n_1\right\rangle_{\partial\Omega_1} + \alpha\frac{\kappa}{h}\left\langle v, u\right\rangle_{\partial\Omega_1} - \kappa\left\langle\frac{\partial v}{\partial x}n_1, u\right\rangle_{\partial\Omega_1} = \left(v, f\right)_{\Omega_1}$$

$$+ \alpha\frac{\kappa}{h}\left\langle v, \overline{u}\right\rangle_{\partial\Omega_1} - \kappa\left\langle\frac{\partial v}{\partial x}n_1, \overline{u}\right\rangle_{\partial\Omega_1}$$

where α is a penalty parameter tuned to ensure stability, h is the element size, and $\overline{u}$ is the prescribed Dirchlet function.

Considering the problem of interest in the left sub-domain $\Omega_1 = [0, 0.4]$, and evaluating the boundary terms yields:

$$
\big(v, \frac{\partial u}{\partial t}\big)_{\Omega_1} + \big(\frac{\partial v}{\partial x}, \kappa \frac{\partial u}{\partial x}\big)_{\Omega_1} - \kappa\Big[\big(v\frac{\partial u}{\partial x}n_1\big)|_{x=0.4} - \big(v\frac{\partial u}{\partial x}n_1\big)|_{x=0}\Big] + \alpha\frac{\kappa}{h}\Big[\big(vu\big)|_{x=0.4} - \big(vu\big)|_{x=0}\Big]
$$
$$
- \kappa\Big[\big(\frac{\partial v}{\partial x}n_1 u\big)|_{x=0.4} - \big(\frac{\partial v}{\partial x}n_1 u\big)|_{x=0}\Big] = \big(v, f\big)_{\Omega_1} + \alpha\frac{\kappa}{h}\Big[\big(v\overline{u}\big)|_{x=0.4} - \big(v\overline{u}\big)|_{x=0}\Big]
$$
$$
- \kappa\Big[\big(\frac{\partial v}{\partial x}n_1\overline{u}\big)|_{x=0.4} - \big(\frac{\partial v}{\partial x}n_1\overline{u}\big)|_{x=0}\Big]
$$

Recalling the fact that $n_1 = 1$ at $x = 0.4$ and $n_1 = -1$ at $x = 0$, the previous weak form is simplified to:

$$
\big(v, \frac{\partial u}{\partial t}\big)_{\Omega_1} + \big(\frac{\partial v}{\partial x}, \kappa \frac{\partial u}{\partial x}\big)_{\Omega_1} - \kappa\Big[v\frac{\partial u}{\partial x}(0.4) + v\frac{\partial u}{\partial x}(0)\Big] + \alpha\frac{\kappa}{h}\Big[vu(0.4) - vu(0)\Big]
$$
$$
- \kappa\Big[\frac{\partial v}{\partial x}u(0.4) + \frac{\partial v}{\partial x}u(0)\Big] = \big(v, f\big)_{\Omega_1} + \alpha\frac{\kappa}{h}\Big[v\overline{u}(0.4) - v\overline{u}(0)\Big]
$$
$$
- \kappa\Big[\frac{\partial v}{\partial x}\overline{u}(0.4) + \frac{\partial v}{\partial x}\overline{u}(0)\Big]
$$

Using the FE approximation presented earlier in Section 4.1, where the vector of shape functions $\boldsymbol{N} = [N_0, N_1, N_2]$ and the vector of nodal values $\boldsymbol{U} = [U_0, U_1, U_2]^T$ is used for the polynomial approximation of the solution in Ω_1. We obtain the following discrete problem in sub-domain $\Omega_1 = [0, 0.4]$:

$$
\int_0^{0.4} \boldsymbol{N}^T\boldsymbol{N} \, d\Omega \frac{d\boldsymbol{U}}{dt} + \int_0^{0.4} \kappa\frac{\partial\boldsymbol{N}^T}{\partial x}\frac{\partial\boldsymbol{N}}{\partial x} \, d\Omega \, \boldsymbol{U} - \kappa\Big[\boldsymbol{N}^T\frac{\partial N_2}{\partial x}U_2 + \boldsymbol{N}^T\frac{\partial N_0}{\partial x}U_0\Big] + \alpha\frac{\kappa}{h}\Big[\boldsymbol{N}^T U_2 - \boldsymbol{N}^T U_0\Big]
$$
$$
- \kappa\Big[\frac{\partial\boldsymbol{N}^T}{\partial x}U_2 + \frac{\partial\boldsymbol{N}^T}{\partial x}U_0\Big] = \int_0^{0.4} \boldsymbol{N}^T f \, d\Omega + \alpha\frac{\kappa}{h}\Big[\boldsymbol{N}^T U_2^{(\Omega_2)} - \boldsymbol{N}^T(0)\Big]
$$
$$
- \kappa\Big[\frac{\partial\boldsymbol{N}^T}{\partial x}U_2^{(\Omega_2)} + \frac{\partial\boldsymbol{N}^T}{\partial x}(0)\Big]
$$

where $U_2^{(\Omega_2)}$ is Dirchlet value imposed at the interface and the superscript (Ω_2) indicates that this value is computed by solving the problem in sub-domain Ω_2, $\boldsymbol{M}$ is the mass matrix, $\boldsymbol{K}$ is the stiffness matrix, the vector $\boldsymbol{C} = \boldsymbol{N}^T$ while the vector $\boldsymbol{D} = \frac{\partial\boldsymbol{N}^T}{\partial x}$. It has also been used the fact that $\frac{\partial N_2}{\partial x} = \frac{\partial N_0}{\partial x} = \frac{h}{2}$ since we are using linear elements of equal size. The previous equation is re-written as:

$$
\boldsymbol{M}\frac{d\boldsymbol{U}}{dt} + \boldsymbol{K}\boldsymbol{U} - \kappa\frac{h}{2}\boldsymbol{C}(U_2 + U_0) + \alpha\frac{\kappa}{h}\boldsymbol{C}(U_2 - U_0) - \kappa\boldsymbol{D}(U_2 + U_0) = \boldsymbol{F} + \alpha\frac{\kappa}{h}\boldsymbol{C}U_2^{(\Omega_2)} - \kappa\boldsymbol{D}U_2^{(\Omega_2)}
$$

Further simplification yields:

$$
\boldsymbol{M}\frac{d\boldsymbol{U}}{dt} + \boldsymbol{K}\boldsymbol{U} - \kappa(\frac{h}{2}\boldsymbol{C} - \frac{\alpha}{h} + \boldsymbol{D})U_2 - \kappa(\frac{h}{2}\boldsymbol{C} + \frac{\alpha}{h} + \boldsymbol{D})U_0 = \boldsymbol{F} + (\alpha\frac{\kappa}{h}\boldsymbol{C} - \kappa\boldsymbol{D})U_2^{(\Omega_2)}
$$

Using BDF1 time discretization, the system of equations is written as:

$$
\boldsymbol{M}\frac{\boldsymbol{U}^{n+1} - \boldsymbol{U}^n}{\delta t} + \boldsymbol{K}\boldsymbol{U}^{n+1} - \kappa(\frac{h}{2}\boldsymbol{C} - \frac{\alpha}{h} + \boldsymbol{D})U_2^{n+1} - \kappa(\frac{h}{2}\boldsymbol{C} + \frac{\alpha}{h} + \boldsymbol{D})U_0^{n+1} = \boldsymbol{F} + (\alpha\frac{\kappa}{h}\boldsymbol{C} - \kappa\boldsymbol{D})U_2^{(n+1)(\Omega_2)}
$$

Condition number:

Nitche's method is better conditioned than the standard penalty method. For the heat transfer problem, it is sufficient to take the penalty parameter $\alpha > 2c_i$ to ensure stability where c_i depends on the shape of the elements, so for non-stretched elements $c_i = \mathcal{O}(1)$. It is also known that increasing the value of α leads to a higher condition number, i.e. ill-conditioned system. The advantage of Nitche's method when compared to penalty method is that it allows for lower values of α and thus, better conditioned systems.

5 Coupling in time: Operator splitting techniques

Consider the 1D, transient, convection-diffusion equation:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} + a \frac{\partial u}{\partial x} = f \quad \text{in } [0,1]$$

$$u(x = 0, t) = 0$$
$$u(x = 1, t) = 0$$
$$u(x, t = 0) = 0$$

with $\kappa = 1$, $a = 1$, $f = 1$.

1. **Discretize it in space using finite elements (3 elements) and in time (finite differences, BDF1). Solve the first step of the problem, writing the solution as a function of the time step size δt.**

   - First, the weak form of the problem is obtain by pre-multiplying by a test function $v \in \mathcal{H}^1(\Omega)$ and integrating over the whole domain Ω, then the term involving the 2nd-order spatial derivative is integrated by parts to give the boundary term which is eliminated due to the absence of Neumann boundary conditions:

$$\left(v, \frac{\partial u}{\partial t}\right)_\Omega + \kappa \left(\frac{\partial v}{\partial x}, \frac{\partial u}{\partial x}\right)_\Omega + a\left(v, \frac{\partial u}{\partial x}\right)_\Omega = \left(v, f\right)_\Omega$$

   - Using Galerkin finite elements, the solution u and the test function v are approximated as:

$$u(x,t) \approx u_h(x,t) = \sum_{i=1} U_i(t) N_i(x) = \boldsymbol{N}\boldsymbol{U}$$

$$v(x,t) \approx v_h(x,t) = \sum_{i=1} V_i(t) N_i(x) = \boldsymbol{V}^T \boldsymbol{N}^T$$

   where $\boldsymbol{U}$ is a column-vector containing the nodal values of u, $\boldsymbol{V}$ is a column-vector containing the nodal values of v, and $\boldsymbol{N}$ is a row-vector of the nodal shape functions. By using the FE approximations into the weak form we obtain:

$$\int_0^1 \boldsymbol{N}^T \boldsymbol{N} \ d\Omega \ \frac{d\boldsymbol{U}}{dt} + \int_0^1 \kappa \frac{\partial \boldsymbol{N}^T}{\partial x} \frac{\partial \boldsymbol{N}}{\partial x} \ d\Omega \ \boldsymbol{U} + \int_0^1 a\boldsymbol{N}^T \frac{\partial \boldsymbol{N}}{\partial x} \ d\Omega \ \boldsymbol{U} = \int_0^1 \boldsymbol{N}^T f \ d\Omega$$

   which results in the following algebric problem:

$$\boldsymbol{M}\frac{d\boldsymbol{U}}{dt} + \boldsymbol{K}\boldsymbol{U} + \boldsymbol{C}\boldsymbol{U} = \boldsymbol{F}$$

   where $\boldsymbol{M}$ is called mass matrix, $\boldsymbol{K}$ is the stiffness matrix, $\boldsymbol{C}$ is the convection matrix and $\boldsymbol{F}$ is the right hand side forcing vector.

   The time discretization using BDF1 yields the following system of equations:

$$\boldsymbol{M}\frac{\boldsymbol{U}^{n+1} - \boldsymbol{U}^n}{\delta t} + \boldsymbol{K}\boldsymbol{U}^{n+1} + \boldsymbol{C}\boldsymbol{U}^{n+1} = \boldsymbol{F}^{n+1}$$

   Considering a source term $f = 1$ (not time-dependent), the system of equations is written as:

$$(\frac{1}{\delta t}\boldsymbol{M} + \boldsymbol{K} + \boldsymbol{C})\boldsymbol{U}^{n+1} = \boldsymbol{F} + \frac{1}{\delta t}\boldsymbol{M}\boldsymbol{U}^n$$

Let $\boldsymbol{A} = \frac{1}{\delta t}\boldsymbol{M} + \boldsymbol{K} + \boldsymbol{C}$ and $\boldsymbol{B} = \boldsymbol{F} + \frac{1}{\delta t}\boldsymbol{M}\boldsymbol{U}^n$, the nodal solution at time step $n+1$ is obtained as:

$$\boldsymbol{U}^{n+1} = \boldsymbol{A}^{-1}\boldsymbol{B}$$

- Using a uniform mesh of 3 elements (4 nodes), and imposing the Dirchlet conditions strongly on both ends of the domain ($U_1 = U_4 = 0$) yields a system of size 2×2 where:

$$\boldsymbol{U}^{n+1}_{\text{reduced}} = \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}$$

$$\boldsymbol{A}_{\text{reduced}} = \int_0^1 \left(\frac{1}{\delta t}\begin{bmatrix} N_2 N_2 & N_2 N_3 \\ N_3 N_2 & N_3 N_3 \end{bmatrix} + \begin{bmatrix} \dfrac{\partial N_2}{\partial x}\dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_2}{\partial x}\dfrac{\partial N_3}{\partial x} \\[2mm] \dfrac{\partial N_3}{\partial x}\dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x}\dfrac{\partial N_3}{\partial x} \end{bmatrix} + \begin{bmatrix} N_2\dfrac{\partial N_2}{\partial x} & N_2\dfrac{\partial N_3}{\partial x} \\[2mm] N_3\dfrac{\partial N_2}{\partial x} & N_3\dfrac{\partial N_3}{\partial x} \end{bmatrix} \right) d\Omega$$

$$\boldsymbol{B}_{\text{reduced}} = \int_0^1 \begin{Bmatrix} N_2 \\ N_3 \end{Bmatrix} d\Omega + \frac{1}{\delta t}\int_0^1 \begin{bmatrix} N_2 N_2 & N_2 N_3 \\ N_3 N_2 & N_3 N_3 \end{bmatrix} d\Omega \, \boldsymbol{U}^n$$

where the shape functions $(N_2(x), N_3(x))$ and their derivatives were computed as:

$$N_2 = \begin{cases} 3x & 0 \le x \le 1/3 \\ 2 - 3x & 1/3 \le x \le 2/3 \\ 0 & 2/3 \le x \le 1 \end{cases}, \quad \frac{\partial N_2}{\partial x} = \begin{cases} 3 & 0 \le x \le 1/3 \\ -3 & 1/3 \le x \le 2/3 \\ 0 & 2/3 \le x \le 1 \end{cases}$$

$$N_3 = \begin{cases} 0 & 0 \le x \le 1/3 \\ 3x - 1 & 1/3 \le x \le 2/3 \\ 3 - 3x & 2/3 \le x \le 1 \end{cases}, \quad \frac{\partial N_3}{\partial x} = \begin{cases} 0 & 0 \le x \le 1/3 \\ 3 & 1/3 \le x \le 2/3 \\ -3 & 2/3 \le x \le 1 \end{cases}$$

- Solving the system of equations for the 1st time step using Symbolic Matlab (see appendix A.1) yields:

$$\boldsymbol{U}^1 = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\[2mm] \dfrac{6\,\delta t\,(51\,\delta t + 1)}{2943\,\delta t^2 + 324\,\delta t + 5} \\[4mm] \dfrac{6\,\delta t\,(57\,\delta t + 1)}{2943\,\delta t^2 + 324\,\delta t + 5} \\[4mm] 0 \end{Bmatrix}$$

2. **Solve the same time step by using a first order operator splitting technique.**

- The operator splitting technique consists of defining:

$$\mathcal{L} = \mathcal{L}_a + \mathcal{L}_\nu$$

$$\mathcal{L}_a u = a\frac{\partial u}{\partial x}$$

$$\mathcal{L}_\nu u = -\kappa\frac{\partial^2 u}{\partial x^2}$$

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and therefore, the transient convection-diffusion equation is written as:

$$\frac{\partial u}{\partial t} + \mathcal{L}_{\nu} u + \mathcal{L}_a u = f$$

We introduce the splitting by defining intermediate variables u_a and u_ν. Next, the advance in time is done as follows:

First, we solve for u_a:

$$u_a(t_n) = u^n$$

$$\frac{\partial u_a}{\partial t} + \mathcal{L}_a u_a = 0$$

Second, we solve for u_ν, but the initial condition is $u_a(t_{n+1})$:

$$u_\nu(t_n) = u_a(t_{n+1})$$

$$\frac{\partial u_\nu}{\partial t} + \mathcal{L}_{\nu} u_\nu = f$$

Finally:

$$u^{n+1} = u_\nu(t_{n+1})$$

- The discrete form of this technique is written as:

First, we solve for $\boldsymbol{U}_a$:

$$\boldsymbol{U}_a^n = \boldsymbol{U}^n$$

$$(\frac{1}{\delta t}\boldsymbol{M} + \boldsymbol{C})\boldsymbol{U}_a^{n+1} = \frac{1}{\delta t}\boldsymbol{M}\boldsymbol{U}_a^n$$

Second, we solve for $\boldsymbol{U}_\nu$, but the initial condition is $\boldsymbol{U}_a^{n+1}$:

$$\boldsymbol{U}_\nu^n = \boldsymbol{U}_a^{n+1}$$

$$(\frac{1}{\delta t}\boldsymbol{M} + \boldsymbol{K})\boldsymbol{U}_\nu^{n+1} = \boldsymbol{F} + \frac{1}{\delta t}\boldsymbol{M}\boldsymbol{U}_\nu^n$$

Finally:

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}_\nu^{n+1}$$

- Particularize to the problem at hand:

First, solve: $\boldsymbol{U}_a^1 = (\frac{1}{\delta t}\boldsymbol{M} + \boldsymbol{C})^{-1} \frac{1}{\delta t}\boldsymbol{M}\boldsymbol{U}_a^0 = \boldsymbol{0} \quad \rightarrow \quad$ (because $\boldsymbol{U}^0 = \boldsymbol{0}$)

Second, solve: $\boldsymbol{U}_\nu^1 = (\frac{1}{\delta t}\boldsymbol{M} + \boldsymbol{K})^{-1} \boldsymbol{F} \quad \rightarrow \quad$ (use $\boldsymbol{U}_\nu^0 = \boldsymbol{U}_a^1 = \boldsymbol{0}$)

The solution obtained using Symbolic Matlab (see appendix A.2) is:

$$\boldsymbol{U}^1 = \boldsymbol{U}_\nu^1 = \left\{ \begin{array}{c} 0 \\[4pt] \dfrac{6\ \delta t}{54\ \delta t + 5} \\[10pt] \dfrac{6\ \delta t}{54\ \delta t + 5} \\[10pt] 0 \end{array} \right\}$$

3. **Evaluate the error of the splitting approach with respect to the monolithic approach. Plot the splitting error vs. the time step size for $\delta t = 1$, $\delta t = 0.5$, $\delta t = 0.25$. Comment on the results.**

   - The solution obtained by the monolithic scheme is:

   $$\boldsymbol{U}^1_{\text{reduced,monolithic}} = \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{6\ \delta t\ (51\ \delta t + 1)}{2943\ \delta t^2 + 324\ \delta t + 5} \\[3mm] \dfrac{6\ \delta t\ (57\ \delta t + 1)}{2943\ \delta t^2 + 324\ \delta t + 5} \end{Bmatrix}$$

   - The solution obtained by the operator-splitting scheme is:

   $$\boldsymbol{U}^1_{\text{reduced,split}} = \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{6\ \delta t}{54\ \delta t + 5} \\[3mm] \dfrac{6\ \delta t}{54\ \delta t + 5} \end{Bmatrix}$$

   - The absolute error of the operator-splitting technique is:

   $$\boldsymbol{e} = \begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix} = \boldsymbol{U}^1_{\text{reduced,monolithic}} - \boldsymbol{U}^1_{\text{reduced,split}}$$

   Using $\delta t = [1, 0.5, 0.25]$, the absolute errors are:

δt	1	0.5	0.25
$|e_1|$	6.3e-3	5.7e-3	4.7e-3
$|e_2|$	4.7e-3	4.3e-3	3.7e-3

   - Figure 4 shows the splitting error as a function of δt, where the $\mathcal{L}_\infty$-norm of the error ($|\boldsymbol{e}|_\infty = \max\limits_i \ |e_i|$) is used.
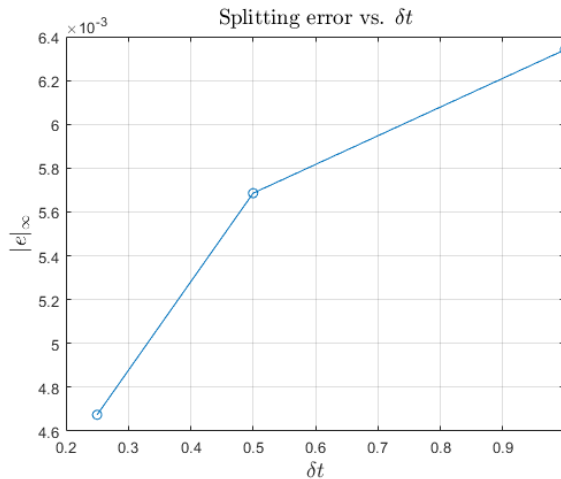

Figure 4: Splitting error as a function of δt

   - The splitting error at different values of δt indicates that this technique converges to the solution of the monolithic scheme by reducing δt. It is actually expected because the operator-splitting technique introduces a splitting error of $\mathcal{O}(\delta t)$.

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6 Coupling in time: Fractional step methods

Consider the fractional step approach for the incompressible Navier-Stokes equations (Yosida scheme):

$$M\frac{1}{\delta t}\left(\hat{U}^{n+1} - U^n\right) + K\hat{U}^{n+1} = f - G\tilde{P}^{n+1} \tag{59a}$$

$$DM^{-1}GP^{n+1} = \frac{1}{\delta t}D\hat{U}^{n+1} - DM^{-1}G\tilde{P}^{n+1} \tag{59b}$$

$$M\frac{1}{\delta t}\left(U^{n+1} - \hat{U}^{n+1}\right) + \alpha K\left(U^{n+1} - \hat{U}^{n+1}\right) + G\left(P^{n+1} - \tilde{P}^{n+1}\right) = 0 \tag{59c}$$

1. **Which is the optimal value for the α parameter?**

   - First, we recall the discrete form of the incompressible Navier-Stokes equations using BDF1 time-integration without decoupling of pressure and velocity:

     $$M\frac{1}{\delta t}\left(U^{n+1} - U^n\right) + KU^{n+1} = f - GP^{n+1} \tag{60a}$$

     $$DU^{n+1} = 0 \tag{60b}$$

   - Summing the two equations (59a) and (59c) yields:

     $$M\frac{1}{\delta t}\left(U^{n+1} - U^n\right) + K\left(\hat{U}^{n+1} + \alpha U^{n+1} - \alpha\hat{U}^{n+1}\right) = f - GP^{n+1}$$

   - By comparing the previous equation with equation (60a), it is observed that we obtain the original scheme of the problem by setting $\alpha = 1$. Therefore, the optimal value is $\alpha = 1$.

2. **What is the source of error of the scheme?**

   - By setting $\alpha = 1$, the sources of error are the approximations $\hat{U}$ and $\hat{P}$ for the velocity and pressure which lead to the relaxation of the incompressibility constraint as given by equation (59b), where the consistent incompressibility constraint should be:

     $$DM^{-1}GP^{n+1} = DM^{-1}f - DM^{-1}KU^{n+1} + \frac{1}{\delta t}DU^n$$

   - For values of α different from 1, the sources of error are the approximations $\hat{U}$ and $\hat{P}$ for the velocity and pressure as well as the value of the parameter α.

7 ALE formulations

7.1 Question 1: Obtaining ALE description from spatial description

Given the spatial description of a property

$$\gamma(x, y, z, t) = [2x, ye^t, z]$$

the equations of movement:

$$x = Xe^t$$
$$y = Y + e^t - 1$$
$$z = Z$$

and the equations of the movement of the mesh:

$$x_m = \mathcal{X} + \alpha t$$
$$y_m = \mathcal{Y} - \beta t$$
$$z_m = \mathcal{Z}$$

(a) To obtain the description of the property γ in terms of the ALE coordinates $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ we substitute the equations of the movement of the mesh into the spatial description of the property:

$$\boxed{\gamma_{\text{ALE}}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, t) = \left[2(\mathcal{X} + \alpha t), (\mathcal{Y} - \beta t)e^t, \mathcal{Z}\right]}$$

(b) The velocity of the particles is:

$$\boxed{\boldsymbol{v} = \frac{\partial \boldsymbol{x}(\boldsymbol{X}, t)}{\partial t} = \left[Xe^t, e^t, 0\right]}$$

The velocity of the mesh is:

$$\boxed{\boldsymbol{v}_{\text{mesh}} = \frac{\partial \boldsymbol{x}(\boldsymbol{\mathcal{X}}, t)}{\partial t} = [\alpha, -\beta, 0]}$$

(c) The material temporal derivative of $\gamma_{\text{ALE}}(\boldsymbol{\mathcal{X}}, t)$ is computed as:

$$\frac{d}{dt}\gamma_{\text{ALE}}(\boldsymbol{\mathcal{X}}, t) = \frac{\partial \gamma_{\text{ALE}}(\boldsymbol{\mathcal{X}}, t)}{\partial t} + \boldsymbol{\nabla}\gamma(\boldsymbol{x}, t) \cdot (\boldsymbol{v} - \boldsymbol{v}_{\text{mesh}})$$

$$= \left\{ \begin{array}{c} 2\alpha \\ (\mathcal{Y} - \beta(1 + t))\, e^t \\ 0 \end{array} \right\} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} Xe^t - \alpha \\ e^t + \beta \\ 0 \end{array} \right\}$$

$$= \left\{ \begin{array}{c} 2\alpha \\ (\mathcal{Y} - \beta(1 + t))\, e^t \\ 0 \end{array} \right\} + \left\{ \begin{array}{c} 2Xe^t - 2\alpha \\ e^{2t} + \beta e^t \\ 0 \end{array} \right\}$$

$$= \left\{ \begin{array}{c} 2Xe^t \\ (\mathcal{Y} - \beta t + e^t)\, e^t \\ 0 \end{array} \right\}$$

Recalling that $x = Xe^t = \mathcal{X} + \alpha t$, therefore, we get $X = (\mathcal{X} + \alpha t) e^{-t}$. This yields the final expression for the material derivative of $\gamma_{\text{ALE}}(\mathcal{X}, t)$ as:

$$\frac{d}{dt}\gamma_{\text{ALE}}(\mathcal{X}, t) = \left\{ \begin{array}{c} 2(\mathcal{X} + \alpha t) \\ (\mathcal{Y} - \beta t + e^t) e^t \\ 0 \end{array} \right\}$$

7.2   Question 2: ALE form of the incompressible Navier-Stokes equations

By defining the velocity vector $\boldsymbol{c} = \boldsymbol{v} - \boldsymbol{v}_{\text{mesh}}$, the momentum conservation equation in ALE form is written as:

$$\frac{\partial \boldsymbol{u}_{\text{ALE}}(\mathcal{X}, t)}{\partial t} + \boldsymbol{c} \cdot \boldsymbol{\nabla} \boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{x}, t) = \rho(\boldsymbol{x}, t)\boldsymbol{b}(\boldsymbol{x}, t)$$

For incompressible flow, the Cauchy stress tensor $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ is defined as:

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = -p(\boldsymbol{x}, t)\boldsymbol{I} + 2\mu\boldsymbol{\nabla}^s \boldsymbol{u}(\boldsymbol{x}, t)$$

Using this definition in the momentum equation yields:

$$\frac{\partial \boldsymbol{u}_{\text{ALE}}(\mathcal{X}, t)}{\partial t} + \boldsymbol{c} \cdot \boldsymbol{\nabla} \boldsymbol{u}(\boldsymbol{x}, t) + \boldsymbol{\nabla} p(\boldsymbol{x}, t) - \mu\boldsymbol{\nabla}^2 \boldsymbol{u}(\boldsymbol{x}, t) = \rho(\boldsymbol{x}, t)\boldsymbol{b}(\boldsymbol{x}, t)$$

On the other hand, the mass conservation equation in ALE form is written as:

$$\frac{\partial \rho_{\text{ALE}}(\mathcal{X}, t)}{\partial t} + \boldsymbol{c} \cdot \boldsymbol{\nabla} \rho(\boldsymbol{x}, t) + \rho(\boldsymbol{x}, t)\boldsymbol{\nabla} \cdot \boldsymbol{u}(\boldsymbol{x}, t) = 0$$

where for incompressible flow, it is simplified to:

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}(\boldsymbol{x}, t) = 0$$

Therefore, the ALE form of the incompressible Navier-Stokes equations is written as:

$$\frac{\partial \boldsymbol{u}_{\text{ALE}}(\mathcal{X}, t)}{\partial t} + \boldsymbol{c} \cdot \boldsymbol{\nabla} \boldsymbol{u}(\boldsymbol{x}, t) + \boldsymbol{\nabla} p(\boldsymbol{x}, t) - \mu\boldsymbol{\nabla}^2 \boldsymbol{u}(\boldsymbol{x}, t) = \rho(\boldsymbol{x}, t)\boldsymbol{b}(\boldsymbol{x}, t)$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{u}(\boldsymbol{x}, t) = 0$$

7.3 Question 3: Methods for defining mesh movement in ALE formulations

First of all, we state the requirements that mesh movement methods must satisfy:

- In some boundaries of the domain, it must follow the movement of the particles in the boundaries (Lagrangian boundary). This means that the mesh displacement is equal to the boundary displacement, i.e. $\boldsymbol{d}_{\text{mesh}} = \boldsymbol{d}_L$ on $\Gamma_{\text{Lagrangian}}$.

- In some boundaries of the domain, it must remain static (Eulerian boundary). This means that the mesh displacement is equal to zero, i.e. $\boldsymbol{d}_{\text{mesh}} = \boldsymbol{0}$ on Γ_{Eulerian}.

- In the interior of the domain, the mesh movement must not lead to excessively distorted elements to avoid the increase in the numerical approximation error.

There are several possibilities for computing the mesh displacement in the interior of the domain such as:

- Solving Poisson problem: $-\boldsymbol{\nabla}{\cdot}\boldsymbol{\nabla}\boldsymbol{d} = \boldsymbol{0}$ in Ω. This approach has the advantage of solving independently for the displacement components which makes it computationally efficient. However, it might give folded elements if the displacements of the mesh are large which will ruin the simulation (due to the appearance of negative Jacobian).

- Solving elasticity problem: $\boldsymbol{K}\boldsymbol{d} = \boldsymbol{0}$ in Ω. This approach yields less distorted elements when compared to the approach of solving Poisson problem. However, it has higher computational cost because of the coupling between the components of the mesh displacement.

However, it is not always possible to avoid mesh distortion. If displacements are too large, it is necessary to re-mesh after a number of time steps. After re-meshing, all the results need to be projected onto the new mesh.

Further methods for mesh movement exist in the literature, for instance, in [3]. The authors identified two basic mesh movement strategies. The first strategy is *mesh regularization* which aims at keeping the computational mesh as regular as possible and avoiding mesh entanglement during the calculation. While the second strategy is *mesh-adaptation* which aims at concentrating elements in zones of steep solution gradient and again a suitable indication of the error is required as a basic input to the re-mesh algorithm. The authors listed three methods for *mesh regulation* which are:

- *Transfinite Mapping Method*: This method was originally designed for creating a mesh on a geometric region with specified boundaries. The general transfinite method describes an approximate surface or volume at a huge number of points. In the 2-D case, the transfinite mapping can be made to exactly model all domain boundaries, and, thus, no geometric error is introduced by the mapping. It induces a very low-cost procedure, since new nodal coordinates can be obtained explicitly once the boundaries of the computational domain have been discretized. The main disadvantage of this methodology is that it imposes restrictions on the mesh topology, as two opposite curves have to be discretized with the same number of elements.

- *Laplacian Smoothing and Variational Methods*: Which is the same as the approach of solving Poisson problem to obtain mesh displacements discussed earlier. This technique has an important drawback: in a non-convex domain, nodes may run outside it. Techniques to fix this issue either increase the computational cost enormously or introduce new terms in the formulation, which are particular to each geometry.

- *Mesh-Smoothing and Simple Interpolations*: In fact, in ALE, it is possible to use any mesh-smoothing algorithm designed to improve the shape of the elements once the topology is fixed. Simple iterative averaging procedures can be implemented where possible. The goal of this method is to minimize both the squeeze and distortion of each element in the mesh. The main advantage of these mesh-regularization methods is that they are both simple and rather general. They can in fact be applied to unstructured meshes consisting of triangular and quadrilateral elements in 2-D, and to tetrahedral, hexahedral, prism, and pyramidal elements in 3-D.

The second strategy of *mesh-adaptation* is concerned with using the ALE description as an adaptive technique even if the physical domain does not evolve in time. It allows to concentrate more elements in areas where the error is larger which is called r-adaptivity. This technique requires an error estimator to control the mesh movement.

8 Fluid-Structure Interaction

8.1 Question 1: Added mass effect

- The issue of added mass effect appear in Fluid-Structure Interaction when the densities of an incompressible fluid and solid object are similar or close to each other. This issue is of particular interest when using partitioned schemes because it does not always converge.

- The added mass effect is particularly challenging in problems such as in biomechanics or modelling of the interaction between body tissues and water where the fluid and solid densities are very similar.

- In order to fix the issue of non-convergence of partitioned schemes due to the added mass effect, relaxation methods are used which aim at weighing the Dirchlet condition applied at the interface of one of the sub-domains to control the instability. This helps to alleviate the added mass effect.

- A widely used relaxation method is the Aitken relaxation scheme which uses the last two iterates in order to approximate the next one.

- Other possibilities to solve the issue include *Steepest Descent Methods* and *Robin-Robin Boundary Conditions*.

8.2 Question 2: Aitken relaxation applied on an iteration-by-subdomains scheme

Considering the 1D, transient, heat transfer equation:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f \quad \text{in } [0,1]$$
$$u(x = 0, t) = \overline{u}_L$$
$$u(x = 1, t) = \overline{u}_R$$
$$u(x, t = 0) = u_0$$

The iteration-by-subdomain scheme based on the Dirchlet-Neumann coupling is written as:

$$\boxed{\begin{aligned} \frac{\partial u_1^{(n+1)(k)}}{\partial t} - \kappa_1 \frac{\partial^2 u_1^{(n+1)(k)}}{\partial x^2} &= f \quad \text{in } \Omega_1 \\ u_1^{(n+1)(k)} &= \overline{u}_L \quad \text{on } \Gamma_1 \\ \kappa_1 \frac{\partial u_1^{(n+1)(k)}}{\partial n} &= \kappa_2 \frac{\partial u_2^{(n+1)(k-1)}}{\partial n} \quad \text{on } \Gamma \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{\partial u_2^{(n+1)(k)}}{\partial t} - \kappa_2 \frac{\partial^2 u_2^{(n+1)(k)}}{\partial x^2} &= f \quad \text{in } \Omega_2 \\ u_2^{(n+1)(k)} &= \overline{u}_R \quad \text{on } \Gamma_2 \\ u_2^{(n+1)(k)} &= u_1^{(n+1)(l)} \quad \text{on } \Gamma \end{aligned}}$$

Taking $l = k - 1$ yields a Jacobi scheme which allows for parallel solve, while $l = k$ yields a Gauss-Seidel scheme with sequential solve.

It is important not to mix between the time step n and the iteration k in each time step.

For each time step $n + 1$, the initial guess for the iterations is set to be equal to the solution from the previous time step n. i.e. $u^{(n+1)(0)} = u^n$.

The Aitken relaxation method uses the last two iterates in order to approximate the next one. The first iteration is bit tricky in order to obtain two previous iterates. For simplicity, we will write 2 iterations starting from iteration 2.

Using Gauss-Seidel scheme and Aitken relaxation, the 2nd iteration at time step $n+1$ is written as:

$$
\boxed{
\begin{aligned}
&\frac{\partial u_1^{(n+1)(2)}}{\partial t} - \kappa_1 \frac{\partial^2 u_1^{(n+1)(2)}}{\partial x^2} = f &&\text{in } \Omega_1 \\
&u_1^{(n+1)(2)} = \overline{u}_L &&\text{on } \Gamma_1 \\
&\kappa_1 \frac{\partial u_1^{(n+1)(2)}}{\partial n} = \kappa_2 \frac{\partial u_2^{(n+1)(1)}}{\partial n} &&\text{on } \Gamma
\end{aligned}
}
$$

$$
\boxed{
\begin{aligned}
&\frac{\partial u_2^{(n+1)(2)}}{\partial t} - \kappa_2 \frac{\partial^2 u_2^{(n+1)(2)}}{\partial x^2} = f &&\text{in } \Omega_2 \\
&u_2^{(n+1)(2)} = \overline{u}_R &&\text{on } \Gamma_2 \\
&u_2^{(n+1)(2)} = u_2^{(n+1)(1)} + w\left(u_1^{(n+1)(2)} - u_2^{(n+1)(1)}\right) &&\text{on } \Gamma \\
&\text{with } w = \frac{u_2^n - u_2^{(n+1)(1)}}{u_2^n - u_2^{(n+1)(1)} + u_1^{(n+1)(2)} - u_1^{(n+1)(1)}}
\end{aligned}
}
$$

where it has been used that $u_2^{(n+1)(0)} = u_2^n$.

The 3rd iteration at time step $n+1$ is written as:

$$
\boxed{
\begin{aligned}
&\frac{\partial u_1^{(n+1)(3)}}{\partial t} - \kappa_1 \frac{\partial^2 u_1^{(n+1)(3)}}{\partial x^2} = f &&\text{in } \Omega_1 \\
&u_1^{(n+1)(3)} = \overline{u}_L &&\text{on } \Gamma_1 \\
&\kappa_1 \frac{\partial u_1^{(n+1)(3)}}{\partial n} = \kappa_2 \frac{\partial u_2^{(n+1)(2)}}{\partial n} &&\text{on } \Gamma
\end{aligned}
}
$$

$$
\boxed{
\begin{aligned}
&\frac{\partial u_2^{(n+1)(3)}}{\partial t} - \kappa_2 \frac{\partial^2 u_2^{(n+1)(3)}}{\partial x^2} = f &&\text{in } \Omega_2 \\
&u_2^{(n+1)(3)} = \overline{u}_R &&\text{on } \Gamma_2 \\
&u_2^{(n+1)(3)} = u_2^{(n+1)(2)} + w\left(u_1^{(n+1)(3)} - u_2^{(n+1)(2)}\right) &&\text{on } \Gamma \\
&\text{with } w = \frac{u_2^{(n+1)(1)} - u_2^{(n+1)(2)}}{u_2^{(n+1)(1)} - u_2^{(n+1)(2)} + u_1^{(n+1)(3)} - u_1^{(n+1)(2)}}
\end{aligned}
}
$$

8.3 Question 3: Monolithic solver with Dirchlet boundary conditions applied using Lagrange multipliers

Considering the 1D, transient, heat transfer equation:

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} &= f \quad \text{in } [0,1] \\
u(x = 0, t) &= \overline{u}_L \\
u(x = 1, t) &= \overline{u}_R \\
u(x, t = 0) &= u_0
\end{aligned}
$$

Following the same procedure done in section 4.1 and using BDF1 time discretization, it yields the following system of equations:

$$
(\frac{1}{\delta t}\boldsymbol{M} + \boldsymbol{K})\boldsymbol{U}^{n+1} = \boldsymbol{F}^{n+1} + \frac{1}{\delta t}\boldsymbol{M}\boldsymbol{U}^n
$$

Let $\boldsymbol{A} = \frac{1}{\delta t}\boldsymbol{M} + \boldsymbol{K}$ and $\boldsymbol{B} = \boldsymbol{F}^{n+1} + \frac{1}{\delta t}\boldsymbol{M}\boldsymbol{U}^n$, the nodal solution at time step $n+1$ is obtained by solving the system:

$$
\boldsymbol{A}\boldsymbol{U}^{n+1} = \boldsymbol{B} \tag{61}
$$

Using a uniform mesh of 4 elements (5 nodes), the global matrix A and vector B are written as:

$$A = \mathsf{A}_e \left(\frac{1}{\delta t} M^e + K^e \right)$$

$$B = \mathsf{A}_e \left(F^{n+1,e} + \frac{1}{\delta t} M^e U^{n,e} \right)$$

The shape functions of the reference element [-1,1] and their derivatives are recalled:

$$N_1^e(\xi) = \frac{1}{2}(1 - \xi), \qquad \frac{\partial N_1^e}{\partial \xi} = \frac{-1}{2}$$

$$N_2^e(\xi) = \frac{1}{2}(1 + \xi), \qquad \frac{\partial N_2^e}{\partial \xi} = \frac{1}{2}$$

The elemental matrices are computed as (It is assumed that $\kappa = f = 1$):

$$M^e = \int_{-1}^{1} \begin{bmatrix} N_1^e N_1^e & N_1^e N_2^e \\ N_2^e N_1^e & N_2^e N_2^e \end{bmatrix} \frac{l^e}{2} \, d\xi = \frac{l^e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$K^e = \int_{-1}^{1} \begin{bmatrix} \dfrac{\partial N_1^e}{\partial \xi} \dfrac{\partial N_1^e}{\partial \xi} & \dfrac{\partial N_1^e}{\partial \xi} \dfrac{\partial N_2^e}{\partial \xi} \\[2ex] \dfrac{\partial N_2^e}{\partial \xi} \dfrac{\partial N_1^e}{\partial \xi} & \dfrac{\partial N_2^e}{\partial \xi} \dfrac{\partial N_2^e}{\partial \xi} \end{bmatrix} \frac{2}{l^e} \, d\xi = 4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$F^e = \int_{-1}^{1} \left\{ \begin{array}{c} N_1^e \\ N_2^e \end{array} \right\} \frac{l^e}{2} \, d\xi = \frac{1}{8} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}$$

Since all the elements are exactly the same ($l^e = \frac{1}{4}$), we compute the elemental matrices and vectors once and then assemble into the global system. A time step of size $\delta t = 1$ is used. After assembling the elemental contributions, the global system 61 is written as:

$$\frac{1}{24} \begin{bmatrix} 98 & -95 & 0 & 0 & 0 \\ -95 & 196 & -95 & 0 & 0 \\ 0 & -95 & 196 & -95 & 0 \\ 0 & 0 & -95 & 196 & -95 \\ 0 & 0 & 0 & -95 & 98 \end{bmatrix} \left\{ \begin{array}{c} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \\ U_5^{n+1} \end{array} \right\} = \frac{1}{8} \left\{ \begin{array}{c} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{array} \right\} + \frac{1}{24} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \left\{ \begin{array}{c} U_1^n \\ U_2^n \\ U_3^n \\ U_4^n \\ U_5^n \end{array} \right\} = \left\{ \begin{array}{c} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{array} \right\}$$

By imposing Dirchlet boundary conditions at $x = 0$ and $x = 1$ using Lagrange multipliers, the system of equations is written as:

$$\frac{1}{24} \begin{bmatrix} 98 & -95 & 0 & 0 & 0 & 24 & 0 \\ -95 & 196 & -95 & 0 & 0 & 0 & 0 \\ 0 & -95 & 196 & -95 & 0 & 0 & 0 \\ 0 & 0 & -95 & 196 & -95 & 0 & 0 \\ 0 & 0 & 0 & -95 & 98 & 0 & 24 \\ 24 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 24 & 0 & 0 \end{bmatrix} \left\{ \begin{array}{c} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \\ U_5^{n+1} \\ \lambda_1 \\ \lambda_2 \end{array} \right\} = \left\{ \begin{array}{c} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ \bar{u}_L \\ \bar{u}_R \end{array} \right\}$$

The condition number of the matrix A is computed using Matlab and it is equal to $\boxed{38.3156}$.

8.4 Question 4: Monolithic solver for a domain with regions of different thermal conductivity

Considering the 1D, transient, heat transfer equation given in the previous question with a variation in the diffusion coefficient κ within the domain [0,1]. The schematic in Figure 5 explains the problem of interest.
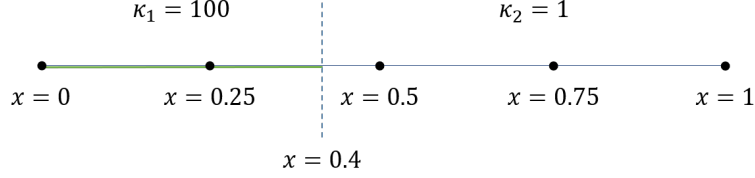

$$\kappa_1 = 100 \qquad \kappa_2 = 1$$

$$x = 0 \qquad x = 0.25 \qquad x = 0.5 \qquad x = 0.75 \qquad x = 1$$

$$x = 0.4$$

Figure 5: The 1D domain [0,1] with varying diffusion coefficient κ

Relating this problem to the previous problem, we note that the elemental mass matrices and forcing vectors remains unchanged for all the elements. The elemental stiffness matrices for elements 3 and 4 also remain unchanged as κ doesn't change while the elemental stiffness matrix for elements 1 is multiplied by 100 as κ is changed from 1 to 100.

$$\boldsymbol{M}^1 = \boldsymbol{M}^2 = \boldsymbol{M}^3 = \boldsymbol{M}^4 = \frac{1}{24}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\boldsymbol{F}^1 = \boldsymbol{F}^2 = \boldsymbol{F}^3 = \boldsymbol{F}^4 = \frac{1}{8}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\boldsymbol{K}^3 = \boldsymbol{K}^4 = 4\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{K}^1 = 400\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The stiffness matrix for element 2 needs to be computed taking into account the variation of κ within the element. This is done by dividing element 2 into two regions $x \in [0.25, 0.4]$ and $x \in [0.4, 0.5]$ which are equivalent to the two regions $\xi \in [-1, 0.2]$ and $\xi \in [0.2, 1]$ in the reference element. Next the stiffness matrix of element 2 is computed as follows:

$$\boldsymbol{K}^2 = \int_{-1}^{1} \kappa \begin{bmatrix} \dfrac{\partial N_1^e}{\partial \xi}\dfrac{\partial N_1^e}{\partial \xi} & \dfrac{\partial N_1^e}{\partial \xi}\dfrac{\partial N_2^e}{\partial \xi} \\[2ex] \dfrac{\partial N_2^e}{\partial \xi}\dfrac{\partial N_1^e}{\partial \xi} & \dfrac{\partial N_2^e}{\partial \xi}\dfrac{\partial N_2^e}{\partial \xi} \end{bmatrix} \frac{2}{l^e}\, d\xi$$

$$= 2\kappa_1 \int_{-1}^{0.2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi + 2\kappa_2 \int_{0.2}^{1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi$$

$$= 240 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{8}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= -241.6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assembling all the elemental contributions $\boldsymbol{K}^e$ yields the global stiffness matrix as:

$$
\boldsymbol{K} = \begin{bmatrix}
400 & -400 & 0 & 0 & 0 \\
-400 & 641.6 & -241.6 & 0 & 0 \\
0 & -241.6 & 245.6 & -4 & 0 \\
0 & 0 & -4 & 8 & -4 \\
0 & 0 & 0 & -4 & 4
\end{bmatrix}
$$

Using a time step of size $\delta t = 1$, the global system 61 is written as:

$$
\begin{bmatrix}
4801/12 & -9599/24 & 0 & 0 & 0 \\
-9599/24 & 19253/30 & -28987/120 & 0 & 0 \\
0 & -28987/120 & 7373/30 & -95/24 & 0 \\
0 & 0 & -95/24 & 49/6 & -95/24 \\
0 & 0 & 0 & -95/24 & 49/12
\end{bmatrix}
\begin{Bmatrix}
U_1^{n+1} \\
U_2^{n+1} \\
U_3^{n+1} \\
U_4^{n+1} \\
U_5^{n+1}
\end{Bmatrix}
=
\begin{Bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4 \\
B_5
\end{Bmatrix}
$$

where $\begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{Bmatrix} = \dfrac{1}{8} \begin{Bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{Bmatrix} + \dfrac{1}{24} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{Bmatrix} U_1^n \\ U_2^n \\ U_3^n \\ U_4^n \\ U_5^n \end{Bmatrix}$

By imposing Dirchlet boundary conditions at $x = 0$ and $x = 1$ using Lagrange multipliers, the system of equations is written as:

$$
\begin{bmatrix}
4801/12 & -9599/24 & 0 & 0 & 0 & 1 & 0 \\
-9599/24 & 19253/30 & -28987/120 & 0 & 0 & 0 & 0 \\
0 & -28987/120 & 7373/30 & -95/24 & 0 & 0 & 0 \\
0 & 0 & -95/24 & 49/6 & -95/24 & 0 & 0 \\
0 & 0 & 0 & -95/24 & 49/12 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
U_1^{n+1} \\
U_2^{n+1} \\
U_3^{n+1} \\
U_4^{n+1} \\
U_5^{n+1} \\
\lambda_1 \\
\lambda_2
\end{Bmatrix}
=
\begin{Bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4 \\
B_5 \\
\overline{u}_L \\
\overline{u}_R
\end{Bmatrix}
$$

The condition number of the matrix $\boldsymbol{A}$ is computed using Matlab and it is equal to $\boxed{4696.8}$. This leads to an ill-conditioned system of equations as the condition number of matrix $\boldsymbol{A}$ is too large.

A Developed codes

A.1 Script for section 5 - question 1

```matlab
% Operator splitting techniques
%
% Question 1:

syms x dt N2 N3;

% Initial U
U_0 = [0; 0];

% from x=0 to x=1/3
N2 = 3*x;
N3 = 0;
N = [N2 N3];
dN = [3 0];
M = N'.*N;
K = dN.'*dN;
C = N'.*dN;
A1 = int(1/dt*M + K + C, x, 0, 1/3);
B1 = int(N', x, 0, 1/3) + int(1/dt*M, x, 0, 1/3)*U_0;

% from x=1/3 to x=2/3
N2 = 2-3*x;
N3 = 3*x-1;
N = [N2 N3];
dN = [-3 3];
M = N'.*N;
K = dN.'*dN;
C = N'.*dN;
A2 = int(1/dt*M + K + C, x, 1/3, 2/3);
B2 = int(N', x, 1/3, 2/3) + int(1/dt*M, x, 1/3, 2/3)*U_0;

% from x=2/3 to x=1
N2 = 0;
N3 = 3-3*x;
N = [N2 N3];
dN = [0 -3];
M = N'.*N;
K = dN.'*dN;
C = N'.*dN;
A3 = int(1/dt*M + K + C, x, 2/3, 1);
B3 = int(N', x, 2/3, 1) + int(1/dt*M, x, 2/3, 1)*U_0;

% Global system
A = A1 + A2 + A3;
B = B1 + B2 + B3;

% System solve
U_1 = A\B;
```

A.2   Script for section 5 - question 2

```matlab
% Operator splitting techniques
%
% Question 2:

syms x dt N2 N3;

% Initial U
U_0 = [0; 0];

% from x=0 to x=1/3
N2 = 3*x;
N3 = 0;
N = [N2 N3];
dN = [3 0];
M = N'.*N;
K = dN.'*dN;
C = N'.*dN;
A1 = int(1/dt*M + K, x, 0, 1/3);
B1 = int(N', x, 0, 1/3);

% from x=1/3 to x=2/3
N2 = 2-3*x;
N3 = 3*x-1;
N = [N2 N3];
dN = [-3 3];
M = N'.*N;
K = dN.'*dN;
C = N'.*dN;
A2 = int(1/dt*M + K, x, 1/3, 2/3);
B2 = int(N', x, 1/3, 2/3);

% from x=2/3 to x=1
N2 = 0;
N3 = 3-3*x;
N = [N2 N3];
dN = [0 -3];
M = N'.*N;
K = dN.'*dN;
C = N'.*dN;
A3 = int(1/dt*M + K, x, 2/3, 1);
B3 = int(N', x, 2/3, 1);

% Global system
A = A1 + A2 + A3;
B = B1 + B2 + B3;

% System solve
U_1 = A\B;
```

References

[1] Finite Element Formulation for Beams. [online] Available at: http://www-g.eng.cam.ac.uk/csml/teaching/4d9/4D9_handout2.pdf [Accessed 6 Apr. 2018].

[2] Donea, J., Huerta, A. (2004). *Finite Element Methods for Flow Problems.* Chichester: Wiley, pp.279-283.

[3] Donea, J., Giuliani, S., Halleux, J. P. (1982). *An arbitrary Lagrangian-Eulerian finite element method for transient dynamic fluid-structure interactions.* Computer methods in applied mechanics and engineering, 33(1-3), 689-723.