Advanced Fluid Mechanics — Rupalee Baldota

Home Work - 4 — Manisha Chetry

1] ⓐ Given Stream function:

$$\psi(r,\theta) = Ur^2 \sin(2\theta)$$

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = Ur\,2\cos 2\theta = 2Ur\cos 2\theta$$

$$V_\theta = -\frac{\partial \psi}{\partial r} = -2Ur\sin 2\theta$$

$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}(y/x)$$

$$u = V_r\cos\theta - V_\theta\sin\theta$$

$$= 2Ur\cos 2\theta\cos\theta - 2Ur\sin 2\theta\sin\theta$$

$$= 2Ur[\cos 2\theta\cos\theta - \sin 2\theta\sin\theta]$$

$$\cancel{= 2Ur[\cos^3\theta + \sin^2\theta\cos\theta]}$$

$$= 2Ur\cos\theta \quad \ldots \because \cos A\cos B + \sin A\sin B = \cos(A-B)$$

$$= 2U\sqrt{x^2+y^2}\,\cos\tan^{-1}\left(\frac{y}{x}\right)$$

$$= 2U\sqrt{x^2+y^2} \times \frac{x}{\sqrt{x^2+y^2}}$$

$$\boxed{u = 2Ux} = \frac{\partial \psi}{\partial y}$$

$$v = V_r\sin\theta + V_\theta\cos\theta$$

$$= 2Ur\cos 2\theta\sin\theta - 2Ur\sin 2\theta\cos\theta$$

$$= 2Ur[\cos 2\theta\sin\theta - \sin 2\theta\cos\theta]$$

$$= 2Ur(-\sin\theta)$$

$$\boxed{v = \cancel{2} -2Uy} = -\frac{\partial \psi}{\partial x}$$

Boundary Conditions,

1) at $x=0$ (along Y axis) $u=0$ (symmetry + flow vel.)

$$u|_{x=0} = 0 \times 2 \times U = 0 \rightarrow \text{satisfied.}$$

2) at $y=0$ (along X axis) $v=0$

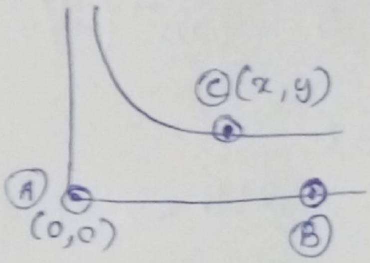
$$v|_{y=0} = -2 \times 0 \times U = 0 \rightarrow \text{satisfied}$$

Here the fluid is assumed to be ideal, hence no slip boundary condition cannot be applied.

$u = 2xU$

$v = -2yU$

$\nabla \times V = 0$ irrotational flow



Here Bernoulli's theorem can be applied between any two points in space.

Between A, C & B,

$$P_0 + 0 = P_x + \frac{1}{2}\rho|\bar{V}|^2$$

$$\therefore P_x = P_0 - \frac{1}{2}\rho\left[(2xU)^2 + (-2yU)^2\right]$$

$$\therefore \boxed{P = P_0 - 2\rho U^2(x^2+y^2)}$$

(b) Navier-Stokes for 2D flows & no body forces:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \text{.... for X}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \text{.... for Y.}$$

$\therefore$ For x,

$$L.H.S. = \rho(0 + (2xU)(2U) + 0)$$
$$= \underline{\underline{4U^2 x \rho}}$$

$$R.H.S. = -\frac{\partial P}{\partial x} + \mu(0+0)$$
$$= 2\rho U^2 (2x)$$ .... taking the derivative of pressure distribution obtained in (a).
$$= \underline{\underline{4\rho U^2 x}}$$

Hence, we get L.H.S = R.H.S.

For Y,

$$L.H.S. = \rho[0 + 0 + (-2yU)(-2U)$$
$$= \underline{\underline{4yU^2\rho}}$$

$$R.H.S. = -\frac{\partial P}{\partial y} + \mu(0+0)$$
$$= 2\rho U^2 (2y)$$
$$= \underline{\underline{4yU^2\rho}}.$$

Hence, L.H.S = R.H.S.

This shows that the former velocities and pressure satisfy the N-S equations.

Now,

The Boundary condition for viscous problem, (along with previous BC) No-slip B.C on $y = 0$,

i.e. $u|_{y=0} = 0$

But, $u|_{y=0} = 2xU \neq 0$ .. so this doesn't satisfy.

(c) $u = 2Uxf'$ ... f is a function only of y

The continuity equation is,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = -2Uf' \Rightarrow \boxed{v = -2Uf}$$

Based on B.Cs.,

(i) No normal velocity on boundary (stationary),

$$v|_{y=0} = 0 \Rightarrow f(0) = 0$$

(ii) No slip at boundary

$$u|_{y=0} = 0 \Rightarrow f'(0) = 0$$

(iii) At $y = \infty$,

$$u|_{y=\infty} = 2xU \Rightarrow f'(\infty) = 1$$

$$v|_{y=\infty} = -2Uf \Rightarrow f(\infty) = y$$

(d) Y-momentum Equation:

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

$\therefore$ Substituting $u = 2xUf'$, $v = -2Uf$

$$2xUf'(0) + (-2Uf)(-2Uf') = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(0 + (-2Uf''))$$

$$\therefore 4U^2ff' = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2U\nu f''$$

$$\therefore 4U^2ff'\rho = \frac{\partial p}{\partial y} - 2U\mu f''$$

$$\therefore \frac{\partial P}{\partial y} = -4\rho U^2 f f' + 2U\mu f'' \quad \text{... this is } f^n \text{ of } y \text{ only.}$$

On integrating we get,

$$\boxed{P(x,y) = -2\rho U\{(f)^2\} - 2\rho U\mu f' + m(x)}$$

We know that $m(x)$ is function of x.
It can be determined by comparing with pressure flow distribution reconsidering large value of y.

$f(y) \to y$ for large value of y.

So P should approach old pressure value.

$$P_0 - 2\rho U^2(x^2+y^2) = -2U^2\rho y^2 - 2U\mu + m(x)$$

(since $f(\infty) = y$ & $f'(\infty) = 1$)

$$m(x) = P_0 - 2\rho U^2 x^2 + 2U\mu$$

$$\therefore \boxed{P(x,y) = P_0 - 2\rho U^2 f^2 + 2\mu U(1-f') - 2\rho U^2 x^2}$$

(e) X - momentum Equation:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\therefore \rho(2xUf')(2Uf') - \rho(2Uf)(2Uxf'') = -\frac{\partial P}{\partial x} + \mu(0 + 2Uxf''')$$

$$\therefore \cancel{\rho(2xU} \; 4\rho x U^2(f')^2 - 4U^2 x\rho f f'' = -2\rho U^2(2x) + 2\mu U x f'''$$

... from pressure calculation.

Cancelling x and dividing by $4\rho U^2$ on all sides

$$(f')^2 - ff'' = 1 + \frac{\nu}{2U} f'''$$

$$\frac{\nu}{2U} f''' + ff'' - (f')^2 + 1 = 0$$

$$\boxed{\frac{\nu}{2U} f''' + ff'' - (f')^2 + 1 = 0}$$

Boundary Condition:

(i) at $u(x,0) = 0 \Rightarrow f'(0) = 0$

(ii) at $v(x,0) = 0 \Rightarrow f(0) = 0$

(iii) at $y \to \infty \Rightarrow f(y) = y \Rightarrow f'(y) = 1$.

The above differential equation can be solved by Runge-Kutta method or so, as it is a 3rd order non-linear ODE in f provided it satisfies the above B.C.s.

(3) Given quadratic form of the velocity profile,

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2 \quad \text{—— (1)}$$

Boundary conditions.

$u(x,0) = 0$

$u(x,\delta) = U$

$\frac{\partial u}{\partial y}(x,\delta) = 0$

Put $y = 0$ in eq^n (1)

$$0 = a + 0 + 0 \Rightarrow a = 0.$$

$$y = \delta. \text{ — (1)} \Rightarrow \frac{U}{U} = 0 + b\frac{\delta}{\delta} + c\left(\frac{\delta}{\delta}\right)^2$$

$$\Rightarrow b + c = 1 \quad \text{—— (2)}$$

Differentiating (1) wrt y.

$$\frac{1}{U}\frac{du}{dy} = 0 + \frac{b}{\delta} + \frac{2cy}{\delta^2}$$

$$0 = 0 + \frac{b}{\delta} + \frac{2c}{\delta} \Rightarrow b + 2c = 0. \quad \text{—— (3)}$$

(2) & (3) $\Rightarrow$ $c = -1$

$b = 2.$

$a = 0, \; b = 2, \; c = -1$

$$\therefore \frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 = 2\eta - \eta^2$$

Now. $$\theta^* = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

$$\theta^* = \delta \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\, d\eta$$

$$= \delta \int_0^1 (2\eta - 4\eta^2 + 2\eta^3 - \eta^2 + 2\eta^3 - \eta^4)\, d\eta$$

$$= \delta \left[\eta^2 - \frac{4}{3}\eta^3 + \frac{1}{2}\eta^4 - \frac{\eta^3}{3} + \frac{1}{2}\eta^4 - \frac{\eta^5}{5}\right]_0^1$$

$$= \delta \left[1 - \frac{4}{3} + 1 - \frac{1}{3} - \frac{1}{5}\right]$$

$$\theta^* = \delta \left[2 - \frac{5}{3} - \frac{1}{5}\right]$$

$$\theta^* = \delta \left[\frac{30 - 25 - 3}{15}\right] = \frac{2\delta}{15}$$

The wall shear stress is given by -

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$\tau_w = \mu \left[\frac{\partial}{\delta \partial \eta}\left\{U(2\eta - \eta^2)\right\}\right]_{\eta=0}$$

$$= \frac{\mu}{\delta} U(2 - 2\eta)\Big|_{\eta=0}$$

$$\therefore \tau_w = \frac{2\mu U}{\delta}$$

But we have momentum integral equation,

$$\frac{d}{dx}(U^2\theta) = \frac{\tau_w}{\rho}$$

$$\frac{2}{15}\frac{d\delta}{dx} = \frac{2\mu U}{\delta\rho U^2}$$

$$\Rightarrow \int \delta\, d\delta = \int \frac{15\mu}{\rho U} dx.$$

$$\Rightarrow \frac{\delta^2}{2} = \frac{15\nu x}{U} + C$$

at leading edge, $x = 0$, $\delta = 0$

$$C = 0$$

$$\therefore \frac{\delta^2}{2} = \frac{15\nu x}{U}$$

$$\Rightarrow \delta^2 = \frac{30\nu x}{U}$$

$$\Rightarrow \delta = 5.4772\sqrt{\frac{\nu x^2}{xU}}$$

$$\Rightarrow \delta = \frac{5.4772x}{\sqrt{Re}}$$

$$\therefore \boxed{\frac{\delta}{x_{(Q)}} = \frac{5.4772}{\sqrt{Re}}}$$ — K.P Approximation for Quadratic Polynomial.

Exact Blasius solution is $\frac{\delta}{x_{(B)}} = \frac{5}{\sqrt{Re}}$

and, cubic Polynomial solution is $\frac{\delta}{x_{(c)}} = \frac{4.64}{\sqrt{Re}}$

$$\therefore \frac{\delta}{x_{(Q)}} > \frac{\delta}{x_{(B)}} > \frac{\delta}{x_{(c)}}.$$