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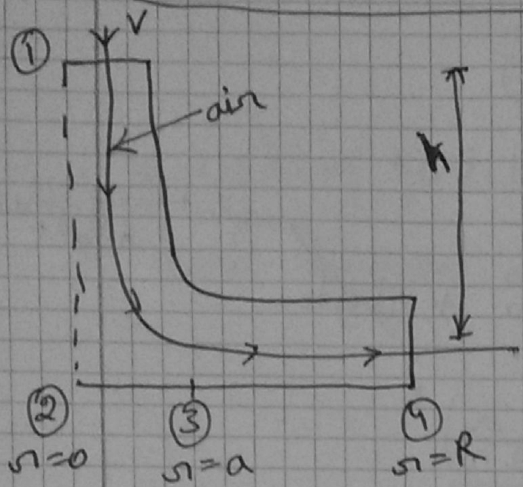
AFM
HOME WORK-2
MSc COMPUTATIONAL MECHANICS
12/11/2015



Taking the following control volume applying mass conservation

$$\frac{DM}{Dt} = \int_{V_t} \frac{\partial \rho}{\partial t}\, dV + \int_{S_t} \rho\, \bar{V}\cdot \bar{n}\, dS$$

assuming steady state

$$\int_{S_t} \rho\, \bar{V}\cdot\bar{n}\, dS = 0$$

taking constant density

$$0 = -\rho V \pi a^2 + 2\rho\pi r h V_r \quad (\because V_r = f(r))$$

$$V_r = \left(\frac{a^2 V}{2h}\right)\frac{1}{r}$$

till now $r \geq a$, $V_r = \frac{C_2}{r}$ where $C_2 = \frac{a^2 V}{2h}$

velocity field is constant at $r=a$

$$\Rightarrow V_r \text{ for } r\leq a\Big|_{r=a} = V_r \text{ for } r\geq a\Big|_{r=a}$$

$$C_1 a = \frac{a^2 V}{2ha}$$

$$\boxed{C_1 = \frac{V}{2h}}$$

applying bernoulli thm between 1 & 2

$$\rho\int_1^2 \cancel{\frac{\partial V}{\partial t}}^{\,0}\cdot dl + P_2 + \frac{1}{2}\rho \cancel{V_2^2} - P_1 - \frac{1}{2}\rho V_1^2 - \cancel{\rho g h} = 0$$

neglecting ($\because \rho_{air} <<$)

0 (stagnation point)

$$P_2 = P_1 + \frac{1}{2}\rho v^2$$

P_1 is P_0 (given).

applying Bernoulli between 2 and 3"

$$P_2 + \frac{1}{2}\rho \underbrace{\cancel{v_2^2}}_{0\ \text{(stagnation)}} + 0 = P_r + \frac{1}{2}\rho v_r^2 + \underbrace{\cancel{\rho g h}}_{\text{neglect (very small height)}}$$

$$P_1 + \frac{1}{2}\rho v^2 = P_r + \frac{1}{2}\rho\left(\frac{v r}{2h}\right)^2$$

$$\boxed{P_r - P_1 = \frac{1}{2}\rho v^2\left(1 - \frac{r^2}{4h^2}\right)} \qquad r \le a$$

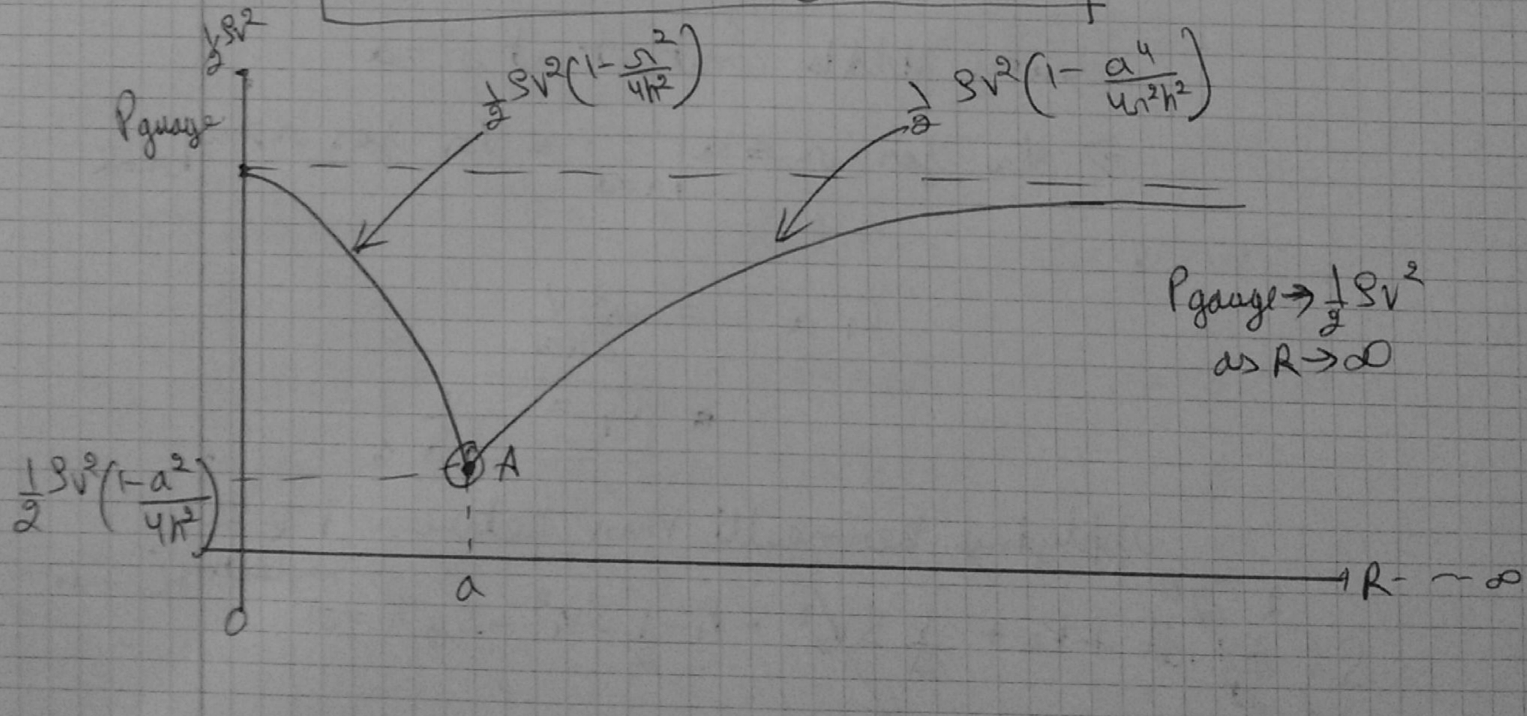
applying bernoulli between 2 & 4.

$$P_2 = P_r + \frac{1}{2}\rho v_r^2$$

$$P_1 + \frac{1}{2}\rho v^2 = P_r + \frac{1}{2}\rho\left(\frac{a^2 v}{2rh}\right)^2$$

$$P_r - P_1 = \frac{1}{2}\rho v^2\left(1 - \frac{a^4}{4h^2r^2}\right)$$

$$\boxed{P_{r,\,\text{gauge}} = \frac{1}{2}\rho v^2\left(1 - \frac{a^4}{4r^2h^2}\right)} \qquad r \ge a$$



Calculating force on cardboard disk

total downward force $= \int_0^R P_r \, 2\pi r \, dr$

$$= \int_0^a \frac{1}{2}\rho v^2\left(1-\frac{r^2}{4h^2}\right)2\pi r\,dr + \int_a^R \frac{1}{2}\rho v^2\left(1-\frac{a^4}{4r^2h^2}\right)2\pi r\,dr$$

$$= \frac{2\pi}{2}\rho v^2\left[\frac{a^2}{\cancel{2}}-\frac{a^4}{16h^2}\right]+\frac{1}{2}\rho v^2(2\pi)\left[\left(\frac{R^2}{2}-\frac{\cancel{a^2}}{\cancel{2}}\right) - \frac{a^4}{4h^2}\ln\left(\frac{R}{a}\right)\right]$$

$$F_{\text{total downward}} = \pi\rho v^2\left[\frac{R^2}{2}-\frac{a^4}{16h^2}\right]+\pi\rho v^2\left(\frac{a^4}{4h^2}\right)\ln\left(\frac{a}{R}\right)$$

$$= \pi\rho v^2\left[\frac{R^2}{2}-\frac{a^4}{16h^2}\right]+\pi\rho v^2\left(\frac{a^4}{4h^2}\right)\ln\left(\frac{a}{R}\right)$$

(can be +ve or −ve) (−ve always, ∵ $a < R$ and rest are +ve).

$\frac{R^2}{2} - \frac{a^4}{16h^2} < 0$ if $R^2 < \frac{a^4}{8h^2}$ or $8h^2R^2 < a^4$

or $\boxed{h < \frac{a^2}{2\sqrt{2}R}}$

F_{down} could be less than zero.

F_{down} = +ve or −ve + (−ve)

in problem

$a = 1\times10^{-2}$ m

$R = 5\times10^{-2}$ m

$h = 0.1\times10^{-2}$ m

$m = 10\times10^{-3}$ Kg

F_{down} + weight of disk = 0

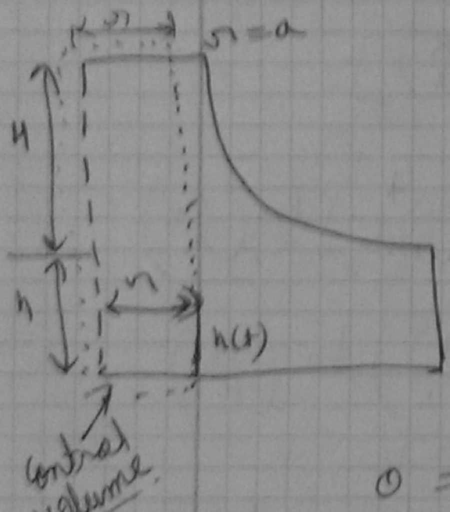
$$1\times10^{-3}\times9.81 + \frac{\pi}{2}\rho v^2\left[(5\times10^{-2})^2 - \frac{(1\times10^{-2})^4}{8(.1\times10^{-2})^2} + \frac{(1\times10^{-2})^4}{2(0.1\times10^{-2})^2}\ln\left(\frac{1\times10^{-2}}{5\times10^{-2}}\right)\right] = 0$$

$$\rho_{air} = 1.225\ kg/m^3$$

$$v^2 = 7.5$$

$$\boxed{v = 2.7386\ m/s}$$

$h = h(t)$; so not a steady state



~~free~~ for $r \leq a$ mass continuity

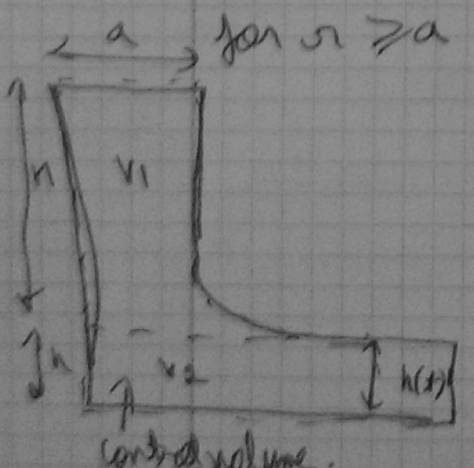
$$\frac{DM}{Dt} = \int_{V_t}\frac{\partial\rho}{\partial t}\,dV + \int_{S_t}\rho\,\bar{V}\cdot\bar{n}\,ds$$

$$0 = \frac{\partial}{\partial t}\left(\rho\pi r^2(h+H)\right) + \rho V_r 2\pi r h - \rho v \pi r^2)$$

ρ is constant & H ~~is so~~ dont change

$$0 = \pi r^2\frac{dh}{dt} + V_r h 2\pi r - v r^2\pi$$

$$\boxed{V_r = \frac{Vr}{2h} - \frac{r}{2h}\frac{dh}{dt} \quad r \leq a}$$



for $r \geq a$ mass continuity.

$$\frac{DM}{Dt} = \int_{V_t}\frac{\partial\rho}{\partial t}\,dV + \int_{St}\rho\,\bar{V}\cdot\bar{n}\,ds$$

$$0 = \frac{\partial}{\partial t}(V_1 + V_2) + 2\pi r h V_r - V\pi a^2$$

$$\frac{\partial V_1}{\partial t} = 0 \ ; \ \frac{\partial V_2}{\partial t} = \pi r^2\frac{dh}{dt}$$

$$\pi r^2 \frac{dh}{dt} = v\pi a^2 - 2\pi r h V_r$$

$$\boxed{V_r = \frac{Va^2}{2\pi h} - \frac{r}{2h}\frac{dh}{dt}} \quad r \geq a$$

using these expressions and applying bernoulli between 2 $(r=0)$ and 4 $(r=R)$

$$\int_0^R \frac{\partial \bar{v}}{\partial t}\cdot d\bar{l} + P_R + \frac{1}{2}\rho V_R^2 - P_2 - \frac{1}{2}\cancel{\rho V_2^2}^{\,0} = 0$$

$$P_0 = P_R$$

$$P_2 = P_1 + \frac{1}{2}\rho v^2 \qquad (P_1 = P_0)$$

$$= (P_0) + \frac{1}{2}\rho v^2$$

$$P_R - P_2 = -\frac{1}{2}\rho v^2 \qquad (P_0 = P_R)$$

$$V_r = \frac{Va^2}{2Rh} - \frac{R}{2h}\left(\frac{dh}{dt}\right)$$

$$\int_0^a \frac{\partial \vec{v}}{\partial t}\cdot d\bar{l} + \int_0^R \frac{\partial \bar{v}}{\partial t}\cdot dl - \frac{1}{2}\rho v^2 + \frac{1}{2}\rho\left[\frac{Va^2}{2Rh} - \frac{R}{2h}\frac{dh}{dt}\right]^2 = 0$$

$$\boxed{\begin{aligned}&\int_0^{=a} \rho\left[\frac{-Vr}{2h^2}\cdot\frac{dh}{dt} + \frac{r}{2h^2}\left(\frac{dh}{dt}\right)^2 - \frac{r}{2h}\frac{d^2h}{dt^2}\right]dr + \\ &\int_a^R \rho\left[-\frac{Va^2}{2rh^2}\frac{dh}{dt} + \frac{r}{2h^2}\left(\frac{dh}{dt}\right)^2 - \frac{r}{2h}\frac{d^2h}{dt^2}\right]dr - \frac{1}{2}\rho v^2 + \\ &\frac{1}{2}\rho\left[\frac{Va^2}{2Rh} - \frac{R}{2h}\frac{dh}{dt}\right]^2 = 0\end{aligned}}$$

Q2

Laplacian in polar coordinates

$$\nabla^2 a = \frac{\partial^2 a}{\partial r^2} + \frac{1}{r}\frac{\partial a}{\partial r} + \frac{1}{r^2}\frac{\partial^2 a}{\partial \theta^2}$$

$$\psi(r, v) = f(r)\sin\theta \text{ with } f(r) = r^2$$

$$\psi = r^2 \sin\theta$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} \quad — (1)$$

$$\frac{\partial \psi}{\partial r} = \alpha r^{\alpha-1}\sin\theta, \quad \frac{\partial^2 \psi}{\partial r^2} = (\alpha^2 - \alpha) r^{\alpha-2}\sin\theta$$

$$\frac{\partial \psi}{\partial \theta} = r^\alpha \cos\theta \quad ; \quad \frac{\partial^2 \psi}{\partial \theta^2} = -r^\alpha \sin\theta$$

Sub in (1)

$$\nabla^2 \psi = (\alpha^2 - \alpha) r^{\alpha-2}\sin\theta + \frac{1}{r}\alpha r^{\alpha-1}\sin\theta + \frac{1}{r^2}\cdot -r^\alpha \sin\theta$$

$$= (\alpha^2 - \alpha) r^{\alpha-2}\sin\theta + \alpha r^{\alpha-2}\sin\theta + r^{\alpha-2}\sin\theta$$

$$\nabla^2 \psi = r^{\alpha-2}\sin\theta\,[\alpha^2 - \alpha + \alpha + 1]$$

$$\Delta \psi = \nabla^2 \psi = r^{\alpha-2}\sin\theta\,[\alpha^2 - 1]$$

as we require plane flow to be irrotational

$$\Delta \psi = 0$$

$$r^{\alpha-2}\sin\theta\,[\alpha^2 - 1] = 0$$

$$\Rightarrow \alpha^2 - 1 = 0 \quad \text{or} \quad \boxed{\alpha = \pm 1} \quad — (4)$$

$$\alpha = \pm 1 \Rightarrow a_1 \frac{\sin\theta}{r} + a_2 r \sin\theta$$

$$\Downarrow$$

$$\psi = a_2 r \sin\theta + a_1 \frac{\sin\theta}{r}$$

$$\therefore \psi = a_2 r \sin\theta + a_1 \frac{\sin\theta}{r} \quad — (2)$$

$$v_x = \sqrt{v_r^2 + v_\theta^2}$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

Scanned by CamScanner

$\cancel{V_\theta = -\frac{\partial \psi}{\partial \theta}}$ $\quad V_\theta = -\frac{\partial \psi}{\partial r}$

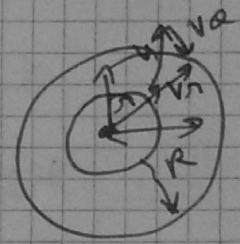
$$V_r = \frac{1}{r}\left(a_1 \frac{\cos\theta}{r} + a_2 r \cos\theta\right)$$

$$\boxed{V_r = \frac{a_1 \cos\theta}{r^2} + a_2 \cos\theta}$$

$$V_\theta = -\left(-\frac{a_1 \sin\theta}{r^2} + a_2 \sin\theta\right)$$

$$V_\theta = \frac{a_1 \sin\theta}{r^2} - a_2 \sin\theta$$

(a)



$r \to \infty$

$V_r = a_2 \cos\theta, \; V_\theta = -a_2 \sin\theta$

$$V = U_x = \sqrt{V_r^2 + V_\theta^2}$$

$$= \sqrt{a_2^2 \cos^2\theta + a_2^2 \sin^2\theta}$$

$$\boxed{U_x = a_2}$$

at $r = R$, $\theta = \pi$; $V_r = 0$; $V_\theta = 0$

$$V_r = -\frac{a_1 \cos\pi}{R^2} + a_2 \cos\pi$$

$$V_r = \frac{-a_1}{R^2} - U_x$$

$$0 = \frac{-a_1}{R^2} - U_x$$

$$\Rightarrow \boxed{a_1 = -U_x R^2}$$

from eq^n (2)

$$\psi = -U_x R^2 \frac{\sin\theta}{r} + U_x r \sin\theta$$

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = -U_x R^2 \frac{\cos\theta}{r^2} + U_x \cos\theta$$

$$V_r = U_x \left[1 - \frac{R^2}{r^2}\right] \cos\theta$$

$$V_\theta = -\frac{\partial \psi}{\partial r} = -U_x \frac{R^2}{r^2}\sin\theta - U_x \sin\theta$$

$$V_\theta = -U_x \sin\theta \left[1 + \frac{R^2}{r^2}\right]$$

at $r = R$

$$V_r = 0 \; ; \; V_\theta = -2U_x\sin\theta$$

Hence, the approx. Boundry conditions.

$$\boxed{r \to \infty \quad \sqrt{\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)^2 + \left(\frac{-\partial\psi}{r}\right)^2} = U_x}$$

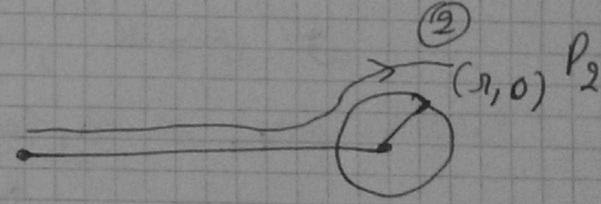
(c) velocity field

$$V = \sqrt{V_r^2 + V_\theta^2}$$

$$V = \sqrt{0 - (2U_x\sin\theta)^2}$$

$$\boxed{V = -2U_x\sin\theta}$$

(d)



applying bernoulli between 1 and 2

$$P_1 = \frac{1}{2}\rho V_x^2 = P_2 + (-2U_x\sin\theta)^2\frac{1}{2}\rho$$

$$P_1 + \frac{1}{2}\rho U^2 = P_2 + \frac{4U^2\sin^2\theta\,\rho}{2}$$

$$P_1 - P_2 = \frac{\cancel{4}^{\,2} U^2 \sin^2\theta\,\rho}{\cancel{2}} - \frac{1}{2}\rho U^2$$

$$P_1 - P_2 = U^2\rho\left[2\sin^2\theta - \frac{1}{2}\right]$$

$$P = \boxed{P_1 - P_2 = U^2\rho\left[\cos 2\theta - \frac{1}{2}\right]}$$

(e) Compute net force acting on cylinder:

$$F_H = \int_0^{2\pi} P R \, d\theta \cos\theta$$

$$F_V = \int_0^{2\pi} P R \, d\theta \sin\theta$$

$$F_H = \int_0^{2\pi} U^2 \rho \left[\cos 2\theta - \frac{1}{2}\right] R \, d\theta \cos\theta$$

$$= U^2 \rho R \left[\underbrace{\int_0^{2\pi} \cos 2\theta \cos\theta \, d\theta}_{\text{integrate by parts}} - \frac{1}{2} \int_0^{2\pi} \cos\theta \, d\theta \right]$$

$$\Rightarrow \int_0^{2\pi} \cos\theta \cos 2\theta \, d\theta = \cos\theta \int_0^{2\pi} \cos 2\theta \, d\theta - \int_0^{2\pi} \frac{\sin\theta \sin 2\theta}{2} \, d\theta$$

$$= \cos\theta \int_0^{2\pi} \cos 2\theta \, d\theta - \left[\sin\theta \int_0^{2\pi} \sin\theta \, d\theta - \int_0^{2\pi} \frac{\cos 2\theta \cos\theta \, d\theta}{2} \right.$$

Let $\int_0^{2\pi} \cos 2\theta \cos\theta = P$

$$= \cos\theta \int_0^{2\pi} \cos 2\theta \, d\theta - \left[\sin\theta \int_0^{2\pi} \sin 2\theta \, d\theta - \frac{P}{2} \right.$$

$$\frac{P}{2} = \cos\theta \int_0^{2\pi} \cos 2\theta \, d\theta - \sin\theta \int_0^{2\pi} \sin 2\theta \, d\theta$$

$$\frac{P}{2} = \left[\frac{\cos\theta \, \cancel{\sin 2\theta}}{2} \right]_0^{2\pi} - \left[\frac{\cancel{\sin\theta} \cos 2\theta}{2} \right]_0^{2\pi} \quad (\sin 2\pi = 0)$$

$(\sin 2\pi = 0)$

$$\boxed{P = 0}$$

$$\therefore F_H = U^2 \rho R \left[0 - \frac{1}{2} \left[\cancel{\sin\theta} \right]_0^{2\pi} \right.$$

$$\boxed{F_H = 0}$$

$$F_V = \int_0^{2\pi} PR \sin\theta \, d\theta = \int_{-\pi}^{\pi} U^2 \rho R \left[\cos 2\theta - \frac{1}{2}\right] \sin\theta \, d\theta$$

$$= U^2 \rho R (-) \int_{\pi}^{0} \left[\cos(-2\theta) - \frac{1}{2}\right] \sin(-\theta) \, d\theta$$

$\int \sin(-\theta) = \int -\sin\theta$

$\int \cos(-\theta) = \int \cos\theta$

$$+ \int_0^{\pi} \left[\cos 2\theta - \frac{1}{2}\right] \sin\theta \, d\theta$$

$$= U^2 \rho R \left[+ \int_0^{\pi} \left[\cos 2\theta - \frac{1}{2}\right](-\sin\theta) + \int_0^{\pi} \left(\cos 2\theta - \frac{1}{2}\right) \sin\theta \, d\theta \right.$$

$$= U^2 \rho R \left[- \int_0^{\pi} \left[\cos\theta - \frac{1}{2}\right] \sin\theta + \int_0^{\pi} \left[\cos 2\theta - \frac{1}{2}\right] \sin\theta \, d\theta \right. \rightarrow 0$$

$$\boxed{F_V = 0}$$

∴ net force on cylinder

$$F_{net} = \sqrt{F_H^2 + F_V^2} = 0$$

$$\boxed{F_{net} = 0}$$