ADVANCED FLUID MECHANICS
Master of Science in Computational Mechanics/Numerical Methods
Fall Semester 2015

Homework 3: Dimensional analysis, compressible flow and Navier-Stokes equations
Due date: December 3, 2015

1. A common engineering challenge faced in pumping viscous crude oil over long distances is the large power consumption required to convey the oil through the pipeline. One proposed solution is to lubricate the pipeline as shown below (figure 1) using a thin layer of an immiscible fluid (such as water) with a lower viscosity to surround the oil and lubricate the motion.

   We shall model the flow as the flow in a cylindrical pipe of radius R with a core of thickness R_1 consisting of very viscous liquid oil with viscosity μ_1 surrounded by a shell of water (or other low viscosity fluid) of thickness $\delta = R - R_1$ that is density matched (so that $\rho_1 = \rho_2 = \rho$ and gravity effects can be neglected) with viscosity $\mu_2 < \mu_1$. The interfacial tension between the two liquids is denoted σ. The average velocity of the oil through the pipe is denoted $\overline{v}_o = Q_{oil}/\pi R_1^2$. Although the oil-water interface shown in the figure above is depicted as

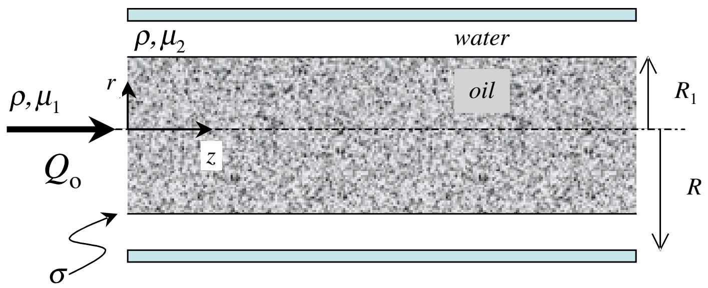

Figure 1: Geometry of a lubricated pipeline.

   planar, in reality under certain operating conditions interfacial waves may form.

   a) Use dimensional analysis to determine an appropriate dimensionless form for expressing the fully-developed pressure drop per unit length in the pipe $-\dfrac{\partial p}{\partial z} = \dfrac{\Delta P}{L}$ as a function of the other relevant parameters in the problem:
   $$\frac{\Delta P}{L} = f(\rho, \overline{v}_o, R, R_1, \mu_1, \mu_2, \sigma).$$
   Use the average oil velocity $\overline{v}_o = Q_{oil}/\pi R_1^2$, the core radius R_1 and the density ρ as your primary variables.

   b) Do you obtain any known dimensionless group? Which one is important in determining whether waves will develop? Express an appropriate inequality on the range for this dimensionless parameter in order for waves <u>not</u> to form.

   c) Note that we have not considered gravity as a driving force to form waves at the interface. Explain whether this hypothesis is reasonable.

Assume that the flow in the pipe remains a perfect smooth core-annular flow, as shown in the sketch. Furthermore, assume that the pressure change across the interface is negligible and pressure gradient is

$$\frac{\partial p}{\partial z} = -\frac{\Delta P}{L} \qquad r \in [0, R]$$

d) The steady-state velocity field is $v = (0, 0, v_z(r))$. Write down Navier-Stokes equations and simplify them accordingly to this velocity field. Clearly state appropriate boundary conditions to be imposed on the wall ($r = R$) and on the interface $r = R_1$.

e) Solve the equations stated in the previous point to obtain an expression for the fully-developed velocity field $v_z(r)$ that are valid in the core domain $0 \leq r \leq R_1$ and the shell $R1 \leq r \leq R$. Which is the interface velocity?
On a single large graph, sketch the velocity profile and the shear stress profile across the entire pipe (i.e. for the region $0 \leq r \leq R$).

f) Find expressions for the volume flow rate of oil Q_o and for the volume flow rate of water Q_w through the pipeline as a function of the imposed pressure ΔP and the other physical parameters defined in the figure.

2. The region $-\infty < x < \ell$ is occupied by a shock tube. The end $x = \ell$ of the tube is closed and the diaphragm is located at $x = 0$. For $0 < x \leq \ell$ the pressure is p_0 and for $-\infty < x < 0$ the pressure is $p_1 (> p_0)$, where $(p_1 - p_0)/p_0 \ll 1$. The velocity is everywhere zero. Draw an x, t diagram showing the wave pattern which results from the breaking of the diaphragm. Obtain expressions for the velocity and pressure behind the wave which is reflected from the closed end of the tube.

Homework3.AFM

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December 2015

Solution1

Part a

By assuming of average oil velocity, core radius and density as the primary variables, we obtain dimensionless groups as follow:

	ρ	$\overline{v}_0$	R_1	R	μ_1	μ_2	σ	$\frac{\Delta P}{L}$
$\mathcal{M}$	1	0	0	0	1	1	1	1
$\mathcal{L}$	-3	1	1	1	-1	-1	0	-2
$\mathcal{T}$	0	-1	0	0	-1	-1	-2	-2

$$\Pi_1 = \rho^a (\overline{v}_0)^b R_1^c R$$
$$\mathcal{M}^0\mathcal{L}^0\mathcal{T}^0 = (\mathcal{M}\mathcal{L}^{-3})^a(\mathcal{L}\mathcal{T}^{-1})^b(\mathcal{L})^c(\mathcal{L})$$
$$a = 0, \quad b = 0, \quad c = -1 \Rightarrow \Pi_1 = \frac{R}{R_1}$$

$$\Pi_2 = \rho^a (\overline{v}_0)^b R_1^c \mu_1$$
$$\mathcal{M}^0\mathcal{L}^0\mathcal{T}^0 = (\mathcal{M}\mathcal{L}^{-3})^a(\mathcal{L}\mathcal{T}^{-1})^b(\mathcal{L})^c(\mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-1})$$
$$a = -1, \quad b = -1, \quad c = -1 \Rightarrow \Pi_2 = \frac{\mu_1}{\rho\overline{v}_0 R_1}$$

and similarly we get,

$$\Pi_3 = \frac{\mu_2}{\rho\overline{v}_0 R_1}$$

$$\Pi_4 = \rho^a (\overline{v}_0)^b R_1^c \sigma$$
$$\mathcal{M}^0\mathcal{L}^0\mathcal{T}^0 = (\mathcal{M}\mathcal{L}^{-3})^a(\mathcal{L}\mathcal{T}^{-1})^b(\mathcal{L})^c(\mathcal{M}\mathcal{T}^{-2})$$
$$a = -1, \quad b = -2, \quad c = -1 \Rightarrow \Pi_4 = \frac{\sigma}{\rho(\overline{v}_0)^2 R_1}$$

$$\Pi_5 = \rho^a (\overline{v}_0)^b R_1^c \frac{\Delta P}{L}$$
$$\mathcal{M}^0\mathcal{L}^0\mathcal{T}^0 = (\mathcal{M}\mathcal{L}^{-3})^a(\mathcal{L}\mathcal{T}^{-1})^b(\mathcal{L})^c(\mathcal{M}\mathcal{L}^{-2}\mathcal{T}^{-2})$$
$$a=-1, \quad b=-2, \quad c=1 \Rightarrow \Pi_5 = \frac{R_1 \Delta P/L}{(\overline{v}_0)^2\rho}$$

Thus,

$$\frac{R_1\Delta P/L}{\rho(\overline{v}_0)^2} = \mathcal{F}\Big(\frac{R}{R_1}, \frac{\mu_1}{\rho \overline{v}_0 R_1}, \frac{\mu_2}{\rho \overline{v}_0 R_1}, \frac{\sigma}{\rho(\overline{v}_0)^2 R_1}\Big)$$

Part b

We have got two well-known dimensionless groups called Reynold number and Weber number. In fact,

$$\frac{1}{\Pi_2} = Re, \quad \frac{1}{\Pi_3} = Re, \quad \frac{1}{\Pi_4} = We$$

Since the interfacial tension try to keep molecules close to each other, it does not let to form wave between two liquids. Besides, the only dimensionless group which has interfacial tension is Weber number. So, Weber number should be as small as it can be. In other words, $We \ll 1$ or $\rho(\overline{v}_0)^2 R_1/\sigma \ll 1$.

Part c

The only driving force to form wave in the pipeline is gravity, but it cannot come to play because there is no difference in density of water and oil in this problem. Therefore, considering gravity as a force to form waves is not reasonable.

Part d

According to velocity field which has given, mass conservation is a trivial equation so we do not need to write it. Navier Stokes equations (momentum conservation) for cylindrical coordinates are:

$$\rho\Big(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\Big) = -\frac{\partial p}{\partial r} + \mu\Big(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\Big) + \rho b_r$$

$$\rho\Big(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\Big) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\Big(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\Big) + \rho b_\theta$$

$$\rho\Big(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial \theta} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\Big) = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z + \rho b_z$$

where,

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

according to $v = (0, 0, v_z(r))$, the first and second equations just tell us that the pressure is not depend on r and θ. we simplify the third Navier Stokes equation as follow:

$$0 = -\frac{\partial p}{\partial z} + \frac{\mu_1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v_z}{\partial r}\Big) \quad (0 \le r \le R_1) \tag{1}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\mu_2}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v_z}{\partial r}\Big) \quad (R_1 \le r \le R) \tag{2}$$

The first equation is for the core domain and the second one is for shell domain.
to obtain the boundary conditions, we should note that on the wall of pipe velocity is 0 and shear stress in center of pipe is 0 too. so,

$$v_z(R) = 0, \quad \tau(0) = 0$$

In addition, since flow in the pipe remains a perfect smooth core-annular flow, we have,

$$\tau(R_1^-) = \tau(R_1^+)$$

for interface velocity we should have,

$$v_z(R_1^-) = v_z(R_1^+)$$

Part e

Since $-\frac{\partial p}{\partial z} = \frac{\Delta P}{L}$ We can write equation (1) and (2) as follow:

$$0 = \frac{\Delta P}{L} + \frac{\mu_1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v_z}{\partial r}\Big) \quad (0 \le r \le R_1) \tag{3}$$

$$0 = \frac{\Delta P}{L} + \frac{\mu_2}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v_z}{\partial r}\Big) \quad (R_1 \le r \le R) \tag{4}$$

then we write (3) by integrating as follow:

$$-\frac{r\Delta P}{\mu_1 L} = \frac{\partial}{\partial r}\Big(r\frac{\partial v_z}{\partial r}\Big) \Rightarrow -\frac{r^2\Delta P}{2\mu_1 L} + A_1 = r\frac{\partial v_z}{\partial r}$$
$$\Rightarrow -\frac{r\Delta P}{2\mu_1 L} + \frac{A_1}{r} = \frac{\partial v_z}{\partial r} \Rightarrow -\frac{r^2\Delta P}{4\mu_1 L} + A_1 \ln r + A_2 = v_z \tag{5}$$

Similarly from (4) we obtain,

$$-\frac{r^2\Delta P}{4\mu_2 L} + B_1 \ln r + B_2 = v_z \tag{6}$$

Equations (5) and (6) are respectively core velocity and shell velocity and A_1, A_2, B_1, B_2 are constants. now, we apply boundary conditions to find these constants.

$$v_z(R) = 0 \Rightarrow -\frac{R^2 \Delta P}{4\mu_2 L} + B_1 \ln R + B_2 = 0 \tag{7}$$

$$\tau(0) = 0 \Rightarrow \mu_1 \frac{\partial v_z}{\partial r}(0) = 0 \Rightarrow -\frac{r\Delta P}{2L} + \frac{\mu_1 A_1}{r} = 0, \quad \text{in } r = 0$$

So the first term should be 0 and to hold the equality, we should have $A_1 = 0$. For interface shear stress we have,

$$-\frac{R_1 \Delta P}{2L} + \frac{\mu_2 B_1}{R_1} = -\frac{R_1 \Delta P}{2L} \Rightarrow \frac{\mu_2 B_1}{R_1} = 0 \Rightarrow B_1 = 0$$

Therefore we can write (7) as follow:

$$-\frac{R^2 \Delta P}{4\mu_2 L} + B_2 = 0 \Rightarrow B_2 = \frac{R^2 \Delta P}{4\mu_2 L}$$

and finally for the interface velocity we have,

$$-\frac{R_1^2 \Delta P}{4\mu_1 L} + A_2 = -\frac{R_1^2 \Delta P}{4\mu_2 L} + \frac{R^2 \Delta P}{4\mu_2 L}$$
$$A_2 = \frac{(R^2 - R_1^2)\Delta P}{4\mu_2 L} + \frac{R_1^2 \Delta P}{4\mu_1 L}$$

Now we get velocity field and shear stress,

$$v_z = \frac{(R^2 - R_1^2)\Delta P}{4\mu_2 L} + \frac{(R_1^2 - r^2)\Delta P}{4\mu_1 L} \quad (0 \leq r \leq R_1)$$

$$v_z = \frac{(R^2 - r^2)\Delta P}{4\mu_2 L} \quad (R_1 \leq r \leq R)$$

$$\tau = -\frac{r\Delta P}{2L} \quad (0 \leq r \leq R_1)$$

$$\tau = -\frac{r\Delta P}{2L} \quad (R_1 \leq r \leq R)$$

The graph has been drawn by hand and it is after solution of part f.

Part f

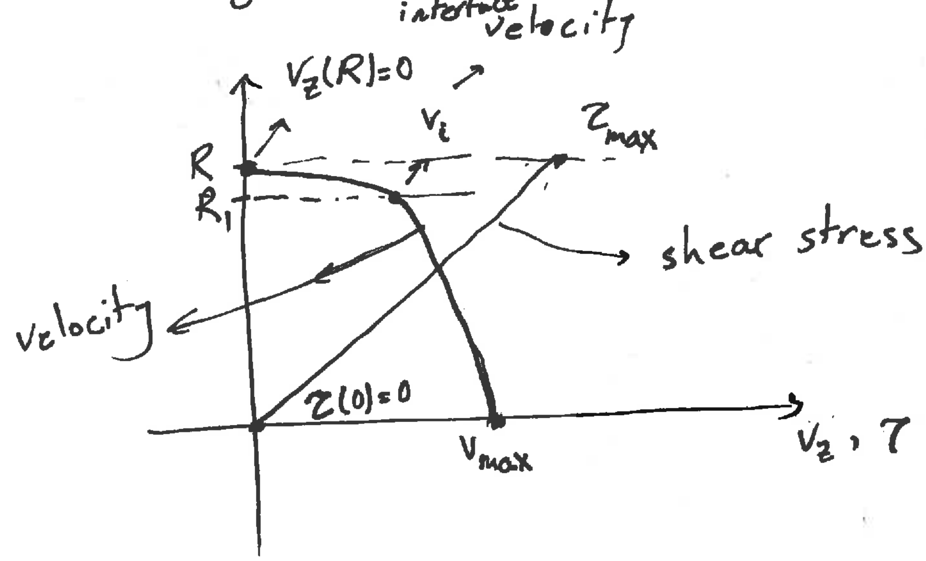
To compute flux of oil Q_o and flux of water Q_w we should integrate of velocity in core surface and shell surface respectively,

$$\begin{aligned}
Q_o &= \int_0^{R_1} \Big(\frac{(R^2-R_1^2)\Delta P}{4\mu_2 L} + \frac{(R_1^2-r^2)\Delta P}{4\mu_1 L}\Big) 2\pi r\ dr \\
&= r^2\pi\frac{(R^2-R_1^2)\Delta P}{4\mu_2 L} + \frac{R_1^2 r^2 \pi}{4\mu_1 L} - r^4\pi\frac{\Delta P}{8\mu_1 L}\Big|_0^{R_1} \\
&= \frac{\pi R_1^2(R^2-R_1^2)\Delta P}{4\mu_2 L} - \frac{\pi R_1^4 \Delta P}{8\mu_1 L}
\end{aligned}$$

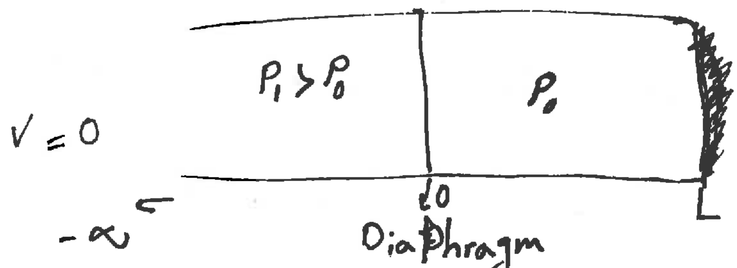
$$\begin{aligned}
Q_w &= \int_{R_1}^{R} \frac{(R^2-r^2)\Delta P}{4\mu_2 L} 2\pi r\ dr = r^2\pi\frac{R^2\Delta P}{4\mu_2 L} - r^4\pi\frac{\Delta P}{8\mu_2 L}\Big|_{R_1}^{R} \\
&= \frac{\pi R^4\Delta P}{4\mu_2 L} - \frac{\pi R^4\Delta P}{8\mu_2 L} - \frac{\pi R_1^2 R^2 \Delta P}{4\mu_2 L} + \frac{\pi R_1^4\Delta P}{8\mu_2 L} \\
&= \frac{\pi R^4\Delta P}{8\mu_2 L} - \frac{2\pi R_1^2 R^2\Delta P}{8\mu_2 L} + \frac{\pi R_1^4\Delta P}{8\mu_2 L} = \frac{\pi(R^2-R_1^2)^2\Delta P}{8\mu_2 L}
\end{aligned}$$

The graph of velocity and shear stress profile:

$\boxed{\text{Part e}}$



Problem 2.

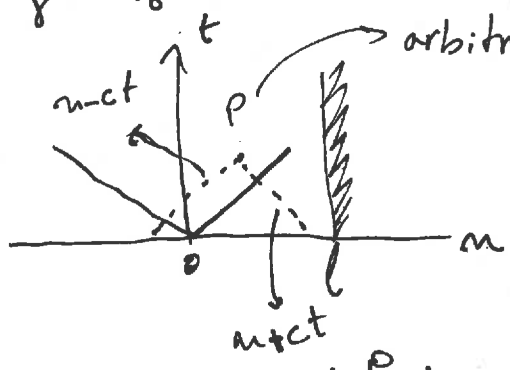


$$\frac{P_1 - P_0}{P_0} \ll 1$$

First we note that rieman invariants are as follow:

$$\frac{u}{c} + \frac{1}{\gamma}\left(\frac{P}{P_0}\right) \quad \text{that is constant along} \quad x - ct = A_1 \rightarrow \text{Constant}$$

$$\frac{u}{c} - \frac{1}{\gamma}\left(\frac{P}{P_0}\right) \quad \text{that is constant along} \quad x + ct = A_2 \rightarrow \text{Constant}$$



$P_0 = P_0 \rightarrow (0 < x < l)$

$P_0 = P_1 \rightarrow (x < 0)$

$v = 0$ everywher

$$\text{For } x+ct \rightarrow \frac{u}{c} - \frac{1}{\gamma}\left(\frac{P}{P_0}\right) = 0 - \frac{1}{\gamma}\left(\frac{P_0}{P_0}\right) \Rightarrow \frac{u}{c} = \frac{1}{\gamma}\left(\frac{P}{P_0} - 1\right) \quad (1)$$

$$\text{For } x-ct \Rightarrow (2)\ \frac{u}{c} + \frac{1}{\gamma}\left(\frac{P}{P_0}\right) = 0 + \frac{1}{\gamma}\left(\frac{P_1}{P_0}\right) \overset{(1)}{\Rightarrow} \frac{1}{\gamma}\left(\frac{P}{P_0} - 1\right) + \frac{1}{\gamma}\left(\frac{P}{P_0}\right) = 0 + \frac{1}{\gamma}\left(\frac{P_1}{P_0}\right)$$

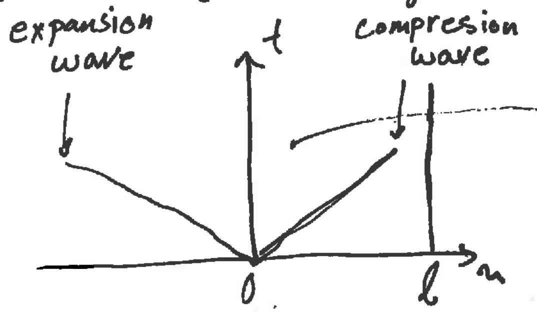
$$\Rightarrow \left(\frac{P-P_0}{P_0}+\frac{P}{P_0}\right)\frac{1}{\cancel{\gamma}}=\frac{1}{\cancel{\gamma}}\frac{P_1}{P_0} \Rightarrow \frac{2P}{\cancel{P_0}}=\frac{P_0+P_1}{\cancel{P_0}}$$

$$\Rightarrow P=\frac{P_0+P_1}{2}$$

left ~~right~~ term

we put the value of P in equation 2. So,

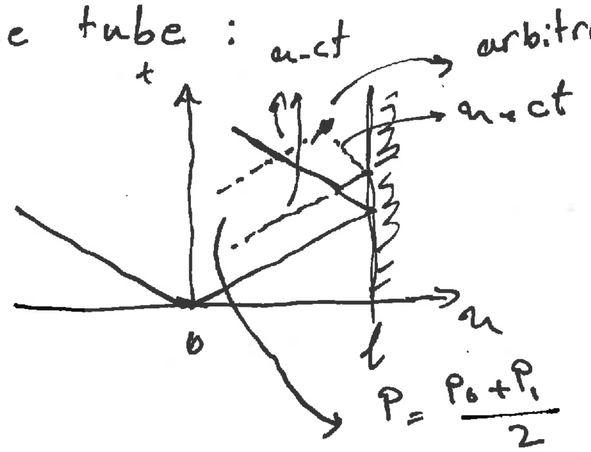
$$\frac{u}{c}-\frac{1}{\gamma}\left(\frac{\frac{P_0+P_1}{2}-P_0}{P_0}\right)=\frac{1}{\gamma}\left(\frac{\frac{P_1-P_0}{2}}{P_0}\right) \Rightarrow \frac{u}{c}=\frac{1}{2\gamma}\left(\frac{P_1}{P_0}-1\right)$$



$$P=\frac{P_0+P_1}{2}$$

$$u=\frac{c}{2\gamma}\left(\frac{P_1}{P_0}-1\right)$$

Obtaining expressions for velocity and pressure behind the wave which is reflected from the closed end of the tube:



$$P=\frac{P_0+P_1}{2}, \quad u=\frac{c}{2\gamma}\left(\frac{P_1}{P_0}-1\right)$$

Velocity of wall is zero. So we just evaluate the pressure on the wall (P_w):

Riemann invariant for $x-ct \rightarrow \frac{u}{c}+\frac{1}{\gamma}\left(\frac{P}{P_0}\right)$

$$\Rightarrow \frac{1}{2\gamma}\left(\frac{P_1}{P_0}-1\right)+\frac{1}{\gamma}\left(\frac{\frac{P_1+P_0}{2}}{P_0}\right)=\frac{1}{\gamma}\left(\frac{P_w}{P_0}\right)$$

$$\Rightarrow \cancel{\frac{1}{\gamma}}\left(\frac{P_1-P_0}{2P_0}+\frac{P_0+P_1}{2P_0}\right)=\left(\frac{P_w}{P_0}\right)\cancel{\frac{1}{\gamma}}$$

$$\Rightarrow \frac{P_1}{\cancel{P_0}}=\frac{P_w}{\cancel{P_0}} \Rightarrow P_w=P_1$$

Now to compute velocity and pressure in the arbitrary point that has been shown in the graph, we should note that the wave is reflected by charactristic lines $x+ct$, $x-ct$

in arbitrary point

$$\text{for } x+ct \Rightarrow \frac{u}{c}-\frac{1}{\gamma}\left(\frac{P^a}{P_0}\right)=-\frac{1}{\gamma}\left(\frac{P_w}{P_0}\right) \Rightarrow \frac{u}{c}=\frac{1}{\gamma}\left(\frac{P^a-P_w}{P_0}\right)$$

$$, P_w=P_1 \Rightarrow \frac{u}{c}=\frac{1}{\gamma}\left(\frac{P^a-P_1}{P_0}\right) \quad (3)$$

$$\text{for } x-ct \Rightarrow \frac{u}{c}+\frac{1}{\gamma}\left(\frac{P^a}{P_0}\right)=\frac{1}{2\gamma}\left(\frac{P_1-P_0}{P_0}\right)+\frac{1}{\gamma}\left(\frac{P_1+P_0}{2P_0}\right)$$

$$\xrightarrow{(3)} \frac{1}{\gamma}\left(\frac{P^a-P_1}{P_0}\right)+\frac{1}{\gamma}\left(\frac{P^a}{P_0}\right)=\frac{1}{2\gamma}\left(\frac{2P_1}{P_0}\right)$$

$$\Rightarrow \cancel{\frac{1}{\gamma}}\left(\frac{2P^a-P_1}{\cancel{P_0}}\right)=\cancel{\frac{1}{\gamma}}\left(\frac{P_1}{\cancel{P_0}}\right) \Rightarrow P^a=P_1$$

$$\Rightarrow \frac{u}{c}+\cancel{\frac{1}{\gamma}\left(\frac{P_1}{P_0}\right)}=\cancel{\frac{1}{\gamma}\left(\frac{P_1}{P_0}\right)}$$

$$\Rightarrow u=0$$

so we have,