HOMEWORK ASSIGNMENT 1

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Contents

Nomenclature

T	Temperature in Kelvin (K).
e	Internal energy (J).
ρ	Density (kg/m^3).
V	Volume (m^3).
v	Specific volume (m^3 /kg) .
s	Entropy (J/K)
Ds/Dt	Material time derivative of entropy
P	Thermodynamic pressure (Pa)
ϕ	Viscous dissipation terms (Joules)
μ	Dynamic viscosity (N s/m^2)
λ	First Lama Constant
q	Heat flux (W/m^2)
$\nabla^s v$	Symmetric part of velocity gradient
$\nabla^{As} v$	Anti-Symmetric part of velocity gradient
k	Coefficient of thermal conductivity (W.m^{-1}.K^{-1})
K	Bulk Modulus (Pascal)

Question 1

1. Let **F** and **G** be two vector functions. Prove the following identities:

Solution:

a)

$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times F) = \mathbf{0}$

$\boldsymbol{\nabla} \cdot \left(\varepsilon_{ijk} F_{k,j}\right) = \varepsilon_{ijk} (F_{k,j}),_i = \varepsilon_{ijk} F_{k,ji} = \underbrace{F_{x,yz} - F_{x,zy}}_{0} + \underbrace{F_{y,zx} - F_{y,xz}}_{0} + + \underbrace{F_{z,xy} - F_{z,yx}}_{0} = 0$

b)

$\nabla \times (\nabla \times F) = \nabla \times \left(\epsilon_{klm} F_{m,l}\right)$

$= \epsilon_{ijk} \epsilon_{klm} F_{m,lj} = \epsilon_{ijk} \epsilon_{lmk} F_{m,lj}$

$= \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}\right) F_{m,lj} = F_{j,ij} - F_{i,jj}$

$= \nabla(\nabla. F) - \nabla^2 F$

c)

$\nabla. (F \times G) = \nabla. \left(\epsilon_{jki} F_j G_k\right)$

$= \epsilon_{jki} F_{j,i} G_k + \epsilon_{jki} F_j G_{k,i}$

$= G_k \epsilon_{kij} F_{j,i} - F_j \epsilon_{jik} G_{k,i}$

$\mathrm{G}. \nabla \times F - \mathrm{F}. \nabla \times G$

Question 2

2. The integral form of the second law of thermodynamics reads

$\frac{D}{Dt}\int_{V_t} \rho s dV \geq -\int_{S_t} \frac{q.n}{T} dS$ ……………(i)

where s is the entropy per unit mass.

Under the following assumptions:

- Newtonian fluid with bulk viscosity (K ≥ 0, μ > 0)
- Fourier's law fpr heat conduction (q = −k∇T, k > 0)

show that above inequality always holds.

Solution:

For a reversible process, entropy change is given by,

$Tds = dQ$

Substitute the expression for dQ, we get

$Tds = de + Pdv$

$$As \qquad v = \frac{1}{\rho}$$

$Tds = de + Pd(\frac{1}{\rho})$

$Tds = de - \frac{Pd\rho}{\rho^2}$

Material rate of change of entropy as the particle flows in time

$T\frac{Ds}{Dt} = \frac{De}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt}$

$\frac{Ds}{Dt} = \frac{1}{T}\frac{De}{Dt} - \frac{P}{T\rho^2}\frac{D\rho}{Dt}$

Taking L.H.S of the given expression (i),

$\frac{D}{Dt}\int \rho s dV$

Applying Reynold Transport theorem,

$= \int_V \rho \frac{Ds}{Dt} dV = \int_V (\frac{\rho}{T}\frac{De}{Dt} - \frac{P}{T\rho}\frac{D\rho}{Dt})dV$

Equation of heat is as follow [1].

$\rho \frac{De}{Dt} = -\nabla . q - P(\nabla . u) + \phi$

After applying continuity equation and equation of heat, we get the following.

$\int_V (\rho \frac{De}{Dt} + \frac{P}{T\rho}\frac{D\rho}{Dt})dV = \int_V [\frac{-\nabla . \mathrm{q}}{T} - \frac{P}{T}(\nabla . \mathrm{u}) + \frac{\phi}{T} + \frac{P}{T}\nabla . \mathrm{u}]\mathrm{dV}$ ………………(1)

Now taking R.H.S of the given formulation and applying divergence law to it.

$\int_s \frac{q.n}{T} dS = \int_V \nabla . \left(\frac{q}{T}\right) dV$ ………………(2)

From (1) and (2), we have

$-\frac{\nabla . \mathrm{q}}{T} - \frac{P}{T}(\nabla . \mathrm{u}) + \frac{\phi}{T} + \nabla . \left(\frac{q}{T}\right) + \frac{P}{T}(\nabla . \mathrm{u}) \geq 0$

$-\frac{\nabla . \mathrm{q}}{T} + \frac{\phi}{T} + \nabla . \left(\frac{q}{T}\right) \geq 0$

$-\frac{\nabla . \mathrm{q}}{T} + \frac{\phi}{T} + \frac{1}{T}\nabla . \mathrm{q} + \mathrm{q}\nabla . (T^{-1}) \geq 0$

As per Fourier law $\mathrm{q} = -\mathrm{k}\nabla \mathrm{T}$

$\frac{\phi}{T} + \frac{k(\nabla T)^2}{T^2} \geq 0$

Above mentioned equation is greater than or equal to 0 because,

1. $\phi \geq 0$

As $\phi = \lambda(\nabla . \boldsymbol{u})^2 + 2\mu \nabla^s \boldsymbol{u} : \nabla \mathbf{u}$

(Where **u** is the velocity vector).

$\lambda(\nabla . \boldsymbol{u})^2 + 2\mu \nabla^s \boldsymbol{u} : (\nabla^s \boldsymbol{u} + \nabla^{As} \boldsymbol{u})$

$\lambda = K - \frac{2}{3}\mu$

$\left(K - \frac{2}{3}\mu\right)(\nabla . \boldsymbol{u})^2 + 2\mu \nabla^s \boldsymbol{u} : \nabla^s \boldsymbol{u}$

$K(\nabla . \boldsymbol{u})^2 - \frac{2}{3}\mu(\nabla . \boldsymbol{u})^2 + 2\mu \nabla^s \boldsymbol{u} : \nabla^s \boldsymbol{u}$

$K(\nabla . \boldsymbol{u})^2 + 2\mu[-\frac{(\nabla . \boldsymbol{u})^2}{3} + \nabla^s \boldsymbol{u} : \nabla^s \boldsymbol{u}]$

Here,

$K(\nabla . \boldsymbol{u})^2 > 0 \; and \; 2\mu > 0$

We are left with proving.

$-\frac{1}{3}(\nabla.\boldsymbol{u})^2 + \nabla^s\boldsymbol{u}:\nabla^s\boldsymbol{u} > 0$

Suppose $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

Above expression will give the following result.

$\frac{1}{2}[u_{12} + u_{21}]^2 - \frac{1}{3}(u_{11} + u_{22})^2 + {u_{11}}^2 + {u_{22}}^2$

For any combinations of the values of $u_{12}, u_{21}, u_{11}, u_{22}$ the above mentioned expression is greater than or equal to zero.

Moreover, if we apply stokes assumption i.e. K=0 to the expression of ϕ ,we get a positive value [1] as shown as follows.

$Deformation\ work\ rate\ per\ volume = \tau_{ij}\frac{\partial \boldsymbol{u}_i}{\partial x_j} = \tau_{ij}e_{ij}$

Upon substituting the Newtonian constitutive equation with stokes assumption,

$\tau_{ij} = -P\delta_{ij} + 2\mu e_{ij} - \frac{2}{3}\mu(\boldsymbol{\nabla}.\boldsymbol{u})\delta_{ij}$,

Relation becomes,

$Deformation\ work = -P(\boldsymbol{\nabla}.\boldsymbol{u}) + 2\mu e_{ij}e_{ij} - \frac{2}{3}\mu(\boldsymbol{\nabla}.\boldsymbol{u})^2$

Where we have used $e_{ij}\delta_{ij} = e_{ii} = \boldsymbol{\nabla}.\boldsymbol{u}$. Denoting the viscous term by ϕ , we obtain,

$Deformation\ work(rate\ per\ volume) = -P(\boldsymbol{\nabla}.\boldsymbol{u}) + \phi$

where $\phi = 2\mu e_{ij}e_{ij} - \frac{2}{3}\mu(\boldsymbol{\nabla}.\boldsymbol{u})^2$. After completing square we get the following

$2\mu[e_{ij} - \frac{1}{3}(\boldsymbol{\nabla}.\boldsymbol{u})\delta_{ij}]^2 \geq 0$

2- In term, $\frac{k\ (\nabla T)^2}{T^2}$

$k > 0$

$\frac{(\nabla T)^2}{T^2} \geq 0$

Hence the given inequality is greater than or equal to zero.

References

[1]. Pijush K. Kundu & Ira M. Cohen, (2015). Fluid Mechanics (Ed. 4). ISBN: 9780123814005.