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Advanced Fluid Mechanics : Homework 4

(1) Consider a flow whose velocity vector coincides with the y axis that impinges on a plane boundary that coincides with the x-axis.

a) Ideal fluid.

$$\psi(r,\theta) = U r^2 \sin(2\theta)$$

Compute v in cartesian coordinates (u, v) and show it verifies boundary conditions. Obtain an expression for the pressure.

$$\frac{\partial \psi}{\partial r} = \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial r} = \cos\theta \frac{\partial \psi}{\partial x} + \sin\theta \frac{\partial \psi}{\partial y} \qquad \begin{pmatrix} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \end{pmatrix}$$

$$\frac{\partial \psi}{\partial \theta} = \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta \frac{\partial \psi}{\partial x} + r\cos\theta \frac{\partial \psi}{\partial y}$$

$$\begin{pmatrix} \frac{\partial \psi}{\partial r} \\ \frac{\partial \psi}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial \psi}{\partial x} \\ \frac{\partial \psi}{\partial y} \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} \frac{\partial \psi}{\partial x} \\ \frac{\partial \psi}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} \frac{\partial \psi}{\partial r} \\ \frac{\partial \psi}{\partial \theta} \end{pmatrix} \qquad \begin{pmatrix} \frac{\partial \psi}{\partial r} = 2Ur\sin(2\theta) \\ \frac{\partial \psi}{\partial \theta} = 2Ur^2\cos(2\theta) \end{pmatrix}$$

$$u = \frac{\partial \psi}{\partial y} = 2Ur\sin(2\theta)\sin(\theta) + 2\cos(\theta)Ur\cos(2\theta) = 2U(\sin(2\theta)y + \cos(2\theta)x)$$

$$v = -\frac{\partial \psi}{\partial x} = -2Ur\sin(2\theta)\cos(\theta) + 2Ur\cos(2\theta)\,\mathrm{sen}(\theta) = 2U(-\sin(2\theta)x + y\cos(2\theta))$$

$$\sin 2\theta = 2\,\mathrm{sen}\theta\cos\theta \rightarrow \sin 2\theta = 2\frac{y}{\sqrt{x^2+y^2}}\cdot\frac{x}{\sqrt{x^2+y^2}} = \frac{2xy}{x^2+y^2}$$

$$\cos 2\theta = \cos^2\theta - \mathrm{sen}^2\theta \rightarrow \cos 2\theta = \frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

$$u = 2U\left(\frac{2xy^2 + x(x^2-y^2)}{x^2+y^2}\right) = 2Ux\left(\frac{x^2+y^2}{x^2+y^2}\right) = 2Ux$$

$$v = 2U\left(\frac{-2xy^2 + y(x^2-y^2)}{x^2+y^2}\right) = 2Uy\left(\frac{-x^2-y^2}{x^2+y^2}\right) = -2Uy$$

$$\bar{V} = (2Ux, -2Uy)$$

Boundary Conditions:

$$\begin{cases} \bar{v}(0,0) = (0,0) & \text{Stagnation Point} \quad \checkmark \\ \bar{v}(0,y) = \bar{v}(y) & \text{Simmetry} \quad \checkmark \\ v(x,0) = 0 & \text{No-penetration} \quad \checkmark \end{cases}$$

Pressure Distribution:

$$\bar{\omega} = \nabla \times \bar{V} = \left(\frac{\partial}{\partial x}(-2Uy) - \frac{\partial}{\partial y}(2Ux)\right)\bar{e}_z = 0 \Rightarrow \text{Irrotational Flow}$$

→ Bernoulli eq. can be apply to any line. (not just streamlines)

$$\cancel{\int_1^2 \frac{\partial v}{\partial t} \cdot ds} + \left(\frac{1}{2}v_2^2 + \frac{P_2}{\rho} - F_2\right) - \left(\frac{1}{2}v_1^2 + \frac{P_1}{\rho} - F_1\right) = 0$$

(stationary)

Being:

1- Stagnation point $(0,0)$ → $v_1 = 0$ $P_1 = P_{stag.}$

2- Any point in the domain → $v_2 = 2U\sqrt{x^2+y^2}$

$$\frac{1}{2}\left(2U\sqrt{x^2+y^2}\right)^2 + \frac{P_2}{\rho} + gy_2 - \frac{P_{stag}}{\rho} + gy_1 = 0$$

Neglecting body forces:

$$\frac{P_{stag} - P_2}{\rho} = 2U^2(x^2+y^2) \rightarrow P_2(x,y) = P_{stag} - 2U^2\rho(x^2+y^2)$$

b) Navier-Stokes equations verification. Boundary conditions verification for the viscous problem.

Navier-Stokes eq:

$$\begin{cases} \nabla \cdot v = 0 \\ \rho(v_t + v \cdot \nabla v) = \nabla \cdot \sigma + \rho b \end{cases}$$

Mass Conservation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow \frac{\partial}{\partial x}(2Ux) + \frac{\partial}{\partial y}(-2Uy) = 2U - 2U = 0 \quad \checkmark$$

$$x \rightarrow \rho\left(\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}}\right) = -\frac{\partial p}{\partial x} + \mu\nabla^2 u + \cancel{\rho b_x}$$

✓ negligible body forces.

$$\rho(2Ux)(2U) = -\frac{\partial(P_{stag} - 2U^2\rho(x^2+y^2))}{\partial x} + \underbrace{\cancel{\mu\nabla^2(2Ux)}}_{0}$$

$$4\rho U^2 x = 4\rho U^2 x \quad \checkmark \quad \text{(x-momentum equation)}$$

$$y \rightarrow \rho\left(\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P}{\partial y} + \cancel{\mu\nabla^2 v} + \cancel{\rho b_y}$$

$$\rho(-2Uy)(-2U) = -\frac{\partial(P_{stag} - 2U^2\rho(x^2+y^2))}{\partial x};$$

$$4\rho U^2 y = 4\rho U^2 y \quad \checkmark \quad \text{(y-momentum equation)}$$

Boundary Conditions for the viscous problem:

$$\begin{cases} \bar{v}(\cancel{x},0) = (0,0) \quad \times \rightarrow u \neq 0 \text{ if } x \neq 0 \\ y \rightarrow \infty \text{ then potential flow is recoverd} \end{cases}$$

No slip condition is not fullfilled

c) Using $u = 2Ux\, f'(y)$, show that the continuity equation requires $v = -2uf(y)$. State appropiate boundary conditions for f.

Continuity equation

$$\nabla \cdot v = 0 \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow 2Uf'(y) + \frac{\partial v}{\partial y} = 0 \rightarrow (v(0)=0)$$

$$\rightarrow v = -2Uf(y)$$

Boundary conditions

$$\bar{v}(\cancel{x},0) = (0,0) \rightarrow \begin{cases} 2Uxf'(0) = 0 \rightarrow f'(0) = 0 \\ -2Uf(\cancel{y}) = 0 \rightarrow f(0) = 0 \end{cases}$$

when $y \rightarrow \infty$, potential flow is recovered

$$y \rightarrow \infty \begin{cases} 2Ux\, f'(y) \rightarrow 2Ux \rightarrow f'(y) = 1 \\ -2Uf(y) \rightarrow -2Uy \rightarrow f(y) = y \end{cases}$$

d) Use the y-momentum equation to obtain an expression for the pressure distribution in terms of f.

Y- Momentum equation:

$$\rho\left(\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\nabla^2 v + \cancel{\rho b_y} \qquad \begin{pmatrix} v = -2Uf(y) \\ u = 2Uxf'(y) \end{pmatrix}$$

$$\rho\left(-2Uf(y)\right)\left(-2Uf'(y)\right) = -\frac{\partial p}{\partial y} + \mu\frac{\partial^2}{\partial y^2}\left(-2Uf(y)\right)$$

$$\rho\, 4U^2 f(y)f'(y) = -\frac{\partial p}{\partial y} + \mu\left(-2Uf''(y)\right)$$

$$\frac{\partial P}{\partial y} = -2U\mu f''(y) - 4U^2\rho\, f(y) f'(y)$$

integrating

$$\int_{P_0(x)}^{P(x,y)} dP = \int_0^y -2U\mu f''(y)\,dy - \int_0^y 4U^2\rho\, f(y) f'(y) \qquad \left(\frac{d(f^2(y))}{dy} = 2f(y)f'(y) \right)$$

$$P(x,y) - P_0(x) = -2U\mu f'(y) - 2U^2\rho f^2(y)$$

if $y \to$ potential flow is recovered

$$P(x,y) = P_{stag} + 2U^2\rho\,(x^2 + y^2)$$

$$\begin{cases} f'(y) = 1 \\ f(y) = y \end{cases}$$

$$P(x,y) = P_{stag} - 2U^2\rho\,(x^2 + \cancel{y^2}) = P_0(x) - 2U\mu - \cancel{2U^2\rho\, y^2} \rightarrow$$

$$\rightarrow P_0(x) = P_{stag} + 2U(\mu - U\rho x^2)$$

So the pressure distribution is:

$$\boxed{P(x,y) = P_{stag} + 2U\mu - 2U^2\rho x^2 - 2U\mu f'(y) - 2U^2\rho f^2(y)}$$

e) Using the x-momentum equation and $P(x,y)$ obtained a differential equation for f. Show that the problem can be solved using BC obtained in point c)

X-momentum equation:

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\nabla^2 u + \cancel{\rho b_x}$$

$$u = 2Ux\, f'(y)$$
$$v = -2U f(y)$$

$$P(x,y) = P_{stag} + 2U\mu\,(1 - f'(y)) - 2U^2\rho\,(f^2(y) + x^2)$$

$$-\frac{\partial p(x,y)}{\partial x} = +\,4U^2\rho x \qquad \nabla^2 u = \cancel{\frac{\partial^2}{\partial x^2}(2Ux f'(y))} + \frac{\partial^2}{\partial y^2}(2Ux f'(y)) = 2Ux f'''(y)$$

$$\downarrow$$

$$\rho\Big((2Ux f'(y))(2U f'(y)) + (-2U f(y))(+2Ux f''(y))\Big) = 4U^2\rho x + \mu 2Ux\, f'''(y)$$

$$4U^2x(f'(y))^2 - 4xU^2 f(y) f''(y) = 4U^2\cancel{\rho}\, x + \frac{2Ux}{\rho} f'''(y)$$

rearranging:

$$f'(y)^2 - f(y) f''(y) = 1 + \frac{1\mu}{2U\rho} f'''(y)$$

$$\frac{\nu}{2U} f'''(y) + f(y) f''(y) - f'(y)^2 + 1 = 0$$

Boundary Conditions

$$f'(0) = f(0) = 0$$

$$y \to \infty \quad \begin{cases} f'(y) = 1 \\ f(y) = y \end{cases}$$

$$\frac{\nu}{2U} f'''(0) + f''(0)\cancel{f(0)} - \cancel{f'(0)^2} + 1 = 0 \rightarrow f'''(0) = -\frac{2U}{\nu}$$

$$\frac{\nu}{2U} f'''(y) + \cancel{f''(y)} f(y) - \underbrace{f'(y)^2}_{1} + 1 = 0 \rightarrow \frac{\nu f'''(y)}{2U} = 0 \rightarrow f'''(y) = 0$$

(2) Use the Kármán-Pohlhausen approximation to compute the boundary layer solution for an uniform flow over a flat plate. Assume a quadratic polynomial form for the velocity profile.

$$\frac{u}{U} = a + \frac{by}{\delta} + c\left(\frac{y}{\delta}\right)^2$$

and use the following B.C:

$$u = 0 \quad \text{at } y = 0$$
$$u = U,\ \frac{\partial u}{\partial y} = 0 \quad \text{at } y = \delta$$

Compare the results with exact Blasius solution and a cubic profile.

- Kármán - Pohlhausen (Quadratic)

$$\frac{u}{U} = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2$$

Applying B.C

$$u(y=0) = 0 \rightarrow \quad 0 = a + b\cdot 0 + c\cdot 0 \rightarrow a = 0$$
$$\left.\begin{array}{ll} u(y=\delta) = U \rightarrow & 1 = b + c \\ \frac{\partial u}{\partial y}(y=\delta) = 0 \rightarrow & 0 = b + 2c \end{array}\right\} \rightarrow \begin{array}{l} b = 2 \\ c = -1 \end{array}$$

$$\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

- Momentum integral equation:

$$\frac{d}{dx}\int_0^\delta u(U-u)\,dy = \frac{\tau_0}{\rho}$$

$$\frac{d}{dx}\int_0^\delta \left(2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)\left(1 - 2\frac{y}{\delta} + \left(\frac{y}{\delta}\right)^2\right)U^2\,dy =$$

$$= U^2\frac{d}{dx}\int_0^\delta \left(-\frac{y^4}{\delta^4} + \frac{4y^3}{\delta^3} - \frac{5y^2}{\delta^2} + \frac{2y}{\delta}\right)dy =$$

$$= U^2 \frac{d}{dx}\left(-\frac{\delta}{5} + \delta - \frac{5}{3}\delta + \delta\right) = U^2 \frac{d}{dx}\left(\frac{2\delta}{15}\right)$$

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu U\left(\frac{2}{\delta} - \frac{2}{\delta^2}y\right)_{y=0} = \frac{2U\mu}{\delta}$$

$$U^2 \frac{d}{dx}\left(\frac{2}{15}\delta\right) = \frac{2U\nu}{\delta}$$

$$\frac{d}{dx}\delta = \frac{15\nu}{\delta U} \quad \rightarrow \text{ integrate :}$$

$$\int_0^\delta \delta\, d\delta = \int_0^x \frac{15\nu}{U}\, dx \rightarrow \frac{\delta^2}{2} = \frac{15\nu x}{U} \qquad (\delta(0) = 0)$$

$$\delta = \sqrt{\frac{30\nu x}{U}} \Rightarrow \delta = 5'477\sqrt{\frac{\nu x}{U}}$$

In adimensional form :

$$\frac{\delta}{x} = 5'477\sqrt{\frac{\nu}{xU}} = \frac{5'477}{\sqrt{Re}}$$

- Kármán - Pohlhausen ((cubic)

$$\frac{u}{U} = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2 + d\left(\frac{y}{\delta}\right)^3$$

B. C :

$\frac{\partial^2 u}{\partial y^2}, u = 0$ at $y = 0$

$u = U, \frac{\partial u}{\partial y} = 0$ at $y = \delta$

$u(0)=0 \rightarrow \quad 0 = a + b\cdot 0 + c\cdot 0 + d\cdot 0 \rightarrow \quad a = 0$

$\frac{\partial^2 u}{\partial y^2}(y=0) = 0 \rightarrow \quad 0 = 2c \quad \rightarrow c = 0$

$$\left.\begin{aligned} u(\delta) = U &\rightarrow \quad 1 = b + d \\ \frac{\partial u}{\partial y}(y=\delta) = 0 &\rightarrow 0 = b + 3d \end{aligned}\right\} \rightarrow \quad \begin{aligned} d &= -1/2 \\ b &= 3/2 \end{aligned}$$

$$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

Momentum integral equation:

$$\frac{d}{dx}\int_0^\delta u(U-u)\,dy = \frac{\tau_0}{\rho}$$

$$\frac{d}{dx}\int_0^\delta \left(\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right)\left(1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right) U^2 dy =$$

$$= U^2 \frac{d}{dx}\int_0^\delta \left(-\frac{1}{4}\frac{y^6}{\delta^6} + \frac{3}{2}\frac{y^4}{\delta^4} - \frac{1}{2}\frac{y^3}{\delta^3} - \frac{9}{4}\frac{y^2}{\delta^2} + \frac{3}{2}\frac{y}{\delta}\right) dy =$$

$$= U^2 \frac{d}{dx}\left[\left(-\frac{1}{28} + \frac{3}{10} - \frac{1}{8} - \frac{9}{12} + \frac{3}{4}\right)\delta\right] = U^2 \frac{d}{dx}\left(\frac{39}{280}\delta\right)$$

$$\frac{\tau_0}{\rho} = \nu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{3}{2}\frac{\nu U}{\delta}$$

$$U^2 \frac{d}{dx}\left(\frac{39}{280}\delta\right) = \frac{3}{2}\frac{\nu U}{\delta} \rightarrow \frac{d\delta}{dx} = \cancel{\frac{140}{13}}\ \frac{140}{13}\frac{\nu}{\delta U}$$ integrating:

$$\int_0^\delta \delta\, d\delta = \int_0^x \frac{140}{13}\frac{\nu}{U}\,dx \rightarrow \frac{\delta^2}{2} = \frac{140}{13}\frac{\nu x}{U} \quad (\delta(0) = 0)$$

$$\delta = 4'641\sqrt{\frac{\nu x}{U}} \rightarrow \frac{\delta}{x} = \frac{4'641}{\sqrt{Re}}$$

Blasius exact solution $\Rightarrow \frac{\delta}{x} = \frac{5}{\sqrt{Re}}$

Comparison:

- Blaius exact $\frac{\delta}{x} = \frac{5}{\sqrt{Re}}$
- Kármán - Pohlhausen (Quadratic) $\frac{\delta}{x} = \frac{5'477}{\sqrt{Re}}$
- Kármán - Pohlhausen (Cubic) $\frac{\delta}{x} = \frac{4'641}{\sqrt{Re}}$