Advanced Fluid Mechanics

Homework4

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Part one

(a) Considering that,

$$r^2 = x^2 + y^2$$
$$\tan(\theta) = \frac{y}{x}$$

We can rewrite the stream function in the following way,

$$\psi(x, y) = \psi(r, \theta) = Ur^2 \sin(2\theta) = 2Ur^2 \sin(\theta)\cos(\theta)$$

Hence, we get the stream function in Cartesian coordinates,

$$\psi(x, y) = 2U(x^2 + y^2)\frac{y}{\sqrt{x^2 + y^2}}\frac{x}{\sqrt{x^2 + y^2}} = 2Uxy$$

$$\text{x} - \text{component: u} = \frac{\partial \psi}{\partial y} = 2Ux$$

$$\text{y} - \text{component: v} = -\frac{\partial \psi}{\partial x} = -2Uy$$

It is clearly shown that at the point (0,0), the velocity is zero, so this point is a stagnation point.

Considering the boundary condition that,

$$\text{u}(\text{x}, 0) = 0$$
$$\text{v}(\text{x}, 0) = 0$$

Sot the x-component velocity does not satisfy the no-slip boundary condition, while the y-component velocity satisfies the boundary condition.

From the Bernoulli equation, the pressure distribution will be,

$$\text{p} = p_0 - 2\rho U^2(x^2 + y^2)$$

where p_0 is the Bernoulli constant that corresponds to the pressure at the stagnation point.

(b) Firstly, we write the N-S equations,

$$\begin{cases} \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \\ \boldsymbol{v_t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} - \nu\boldsymbol{\nabla}^2\boldsymbol{v} + \boldsymbol{\nabla}P = \boldsymbol{0} \end{cases}$$

Substituting the known x,y-component to the N-S equations, we can get the follow equation,

$$\begin{cases} 0 = 0 \\ 4U^2x = 4U^2x \\ 4U^2y = 4U^2y \end{cases}$$

So this velocity and pressure distributions verify the N-S eqautions.

However, the viscous-shear terms in the Navier-Stokers equations are identically zero for the potential-flow fields, so it does not fulfill the boundary conditions for the viscous problem.

(c) Continuity equation,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}=-2Uf'(y)$$

So that the vertical component of the velocity will be of the form,

$$\mathrm{v}=-2\mathrm{Uf(y)}$$

Define the velocity field in this way enable all function f(y) could satisfy the continuity equation. Considering the boundary condition, we stipulate that,

$$\mathrm{f(y)}\rightarrow \mathrm{y} \text{ as } \mathrm{y}\rightarrow\infty$$

$$\mathrm{f}(0)=\mathrm{f}'(0)=0$$

$$\mathrm{f}'(\mathrm{y})\rightarrow 1 \text{ as } \mathrm{y}\rightarrow\infty$$

(d) The y-momentum equation can be written,

$$\mathrm{u}\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2})$$

Substituting the knowing results,

$$4U^2ff'=-\frac{1}{\rho}\frac{\partial p}{\partial y}-2U\nu f''$$

Hence, the pressure distribution,

$$\mathrm{p(x,y)}=-2\rho U^2f^2-2\rho U\nu f'+h(x)$$

where h(x) is some function of x that may be determined by comparison with the potential-flow pressure distribution that should be recovered for large values of y. Since $\mathrm{f(y)}\rightarrow \mathrm{y}$ as $\mathrm{y}\rightarrow\infty$,

$$\mathrm{p(x,y)}\rightarrow -2\rho U^2y^2-2\rho U\nu+h(x)$$

Recalling the in the previous (a), we have,

$$\mathrm{p(x,y)}=p_0-2\rho U^2(x^2+y^2)$$

So compare the two pressure distribution,

$$\mathrm{h(x)}=p_0-2\rho U^2x^2+2\rho U\nu$$

The final pressure distribution should be,

$$\mathrm{p(x,y)}=p_0-2\rho U^2(f^2+x^2)-2\rho U\nu(f'-1)$$

(e) Using the pressure distribution in (d),

$$\frac{\partial p}{\partial x}=-4\rho U^2x$$

So that the x-momentum becomes,

$$4U^2x(f')^2-4U^2xff''=4U^2x+2U\nu xf'''$$

Rewrite it in the following way,

$$\frac{\nu}{2U}f'''+ff''-(f')^2+1=0$$

This is a three order differential equation with the boundary conditons,

$$\mathrm{f(y)}\rightarrow \mathrm{y} \text{ as } \mathrm{y}\rightarrow\infty$$

$$f(0) = f'(0) = 0$$
$$f'(y) \to 1 \text{ as } y \to \infty$$

Part two

According to the boundary conditions:

$$u = 0 \text{ at } y = 0$$
$$u = U, \frac{\partial u}{\partial y} = 0 \text{ at } y = \delta$$

We can solve the quadratic polynomial equation,

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2$$

that,

$$a = 0, b = 2, c = -1$$

Hence, the polynomial equation is,

$$\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

Now in this case is an uniform flow over a flat plate, so the momentum integral equation is,

$$\frac{d}{dx}\int_0^\infty u(U-u)dy = \frac{\tau_0}{\rho}$$

Since,

$$\frac{\tau_0}{\rho} = \nu\frac{\partial u}{\partial y}(y=0) = \frac{2\upsilon U}{\delta}$$

and the momentum thickness,

$$\theta = \int_0^\delta \frac{u}{U}(1-\frac{u}{U})dy = \frac{2}{15}\delta$$

Substituting in the momentum integral equation,

$$\frac{2}{15}U^2\frac{d\delta}{dx} = \frac{2\upsilon U}{\delta}$$

By integration,we obtain,

$$\delta = 5.477\sqrt{\frac{\nu x}{U}}$$

And,

$$\frac{\delta}{x} = \frac{5.477}{\sqrt{Re}}$$

As a comparison, the following table is shown,

	Blasius solution	Quadratic solution	Cubic solution
$thickness: \frac{\delta}{x}$	$\frac{5}{\sqrt{Re}}$	$\frac{5.477}{\sqrt{Re}}$	$\frac{4.64}{\sqrt{Re}}$
$momentum\ thickness: \frac{\theta}{x}$	$\frac{0.644}{\sqrt{Re}}$	$\frac{0.73}{\sqrt{Re}}$	$\frac{0.646}{\sqrt{Re}}$