Homework 3 - AFM

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1 Considering the problem of long distance oil transport proposed in the assignment.

1.1)

The relevant quantities that describe the problem are shown in the table bellow.

	$\frac{\Delta P}{L}[\frac{kg}{m^2 s^2}]$	$\rho[\frac{kg}{m^3}]$	$v_o[\frac{m}{s}]$	$R[m]$	$R_1[m]$	$\mu_1[\frac{kg}{ms}]$	$\mu_2[\frac{kg}{ms}]$	$\sigma[\frac{m}{s^2}]$
M	1	1	0	0	0	1	1	1
L	-2	-3	1	1	1	-1	-1	0
T	-2	0	-1	0	0	-1	-1	-2

With this data, the π-products can be calculated using v_0, ρ and R_1 as primary variables as following:

$$\pi_1 = \frac{\Delta P}{L}\rho^a v_0^b R_1^c \qquad ML^{-2}T^{-2}(ML^{-3})^a(LT^{-1})^b(L)^c = M^0L^0T^0$$

$$\boxed{\pi_1 = \frac{\Delta P}{L}\frac{R_1}{\rho v_0^2}}$$

$$\pi_2 = R\rho^a v_0^b R_1^c \qquad M(ML^{-3})^a(LT^{-1})^b(L)^c = M^0L^0T^0$$

$$\boxed{\pi_2 = \frac{R}{R_1}}$$

$$\pi_3 = \mu_1\rho^a v_0^b R_1^c \qquad ML^{-1}T^{-1}(ML^{-3})^a(LT^{-1})^b(L)^c = M^0L^0T^0$$

$$\boxed{\pi_3 = \frac{\mu_1}{\rho v_0 R_1}}$$

$$\pi_4 = \mu_2\rho^a v_0^b R_1^c \qquad ML^{-1}T^{-1}(ML^{-3})^a(LT^{-1})^b(L)^c = M^0L^0T^0$$

$$\boxed{\pi_4 = \frac{\mu_2}{\rho v_0 R_1}}$$

$$\pi_5 = \sigma\rho^a v_0^b R_1^c \qquad MT^{-2}(ML^{-3})^a(LT^{-1})^b(L)^c = M^0L^0T^0$$

$$\boxed{\pi_5 = \frac{\sigma}{\rho v_0^2 R_1}}$$

1.2)

From the π-products acquired, we can see that π_2, π_3 and π_5 can be related to the Reynolds and Weber numbers as:

$$Re_1 = \frac{1}{\pi_3} \qquad Re_2 = \frac{1}{\pi_4} \qquad We = \frac{1}{\pi_5}$$

Being the Reynolds and Weber numbers:

$$Re = \frac{\rho V l}{\mu} \qquad We = \frac{\rho V^2 l}{\sigma}$$

Where l and V are respectively characteristic length and velocity. The Weber number is used in fluid flows when there is a interface between two different fluids, it measures the relative importance of the inertial forces compared to the surface tension. This quantity is important to determine whether waves will develop on the interface of fluids. In order to this waves not to form the tangential components σ/R_1 has to be bigger them the inertial component $v_0\rho$, in other words, $We < 1$ or $\pi_5 > 1$.

1.3)

The hypothesis can be considered reasonable because both fluids present the same density. Gravity terms become important when density difference are significant, as it can be seem in another dimensionless number called Eotvos Number, which describes the possibility of wave formation in the interface between to fluids.

$$Eo = \frac{\Delta \rho g L^2}{\sigma}$$

For the case presented, $\Delta\rho$ is equal to 0 ($Eo = 0$) preventing the body forces (gravity) being sources of wave generation.

1.4)

The Navier-Stokes equations written in cylindrical coordinates are presented bellow considering a velocity field on the form of $v = (0, 0, v_z(r))$.

$$\rho(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}) = -\frac{\partial p}{\partial r} + \mu(\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}) + \rho b_r$$

$$\rho(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu(\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}) + \rho b_\theta$$

$$\rho(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r\frac{\partial v_z}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \cancel{v_z\frac{\partial v_z}{\partial z}}) = -\frac{\partial p}{\partial z} + \mu(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_z}{\partial r}) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}) + \rho b_z$$

The r and θ components of the velocity, the derivatives v_z with respect to θ and z, gravity terms and the θ and r components of the pressure gradient are neglected (as it is stated in the problem). After the all the simplifications, the resulting equation is presented with its appropriate boundary conditions:

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_z}{\partial r}) = -\frac{1}{\mu}\frac{\Delta P}{L} \\ \\ v_z^w(R) = 0 \\ v_z^w(R_1) = v_z^o(R_1) \\ \tau^w(R_1) = \tau^o(R_1) \\ \tau^w(0) = 0 \end{cases}$$

1.5)

In order to solve this problem, its necessary to integrate the resulting equation twice as following:

$$\frac{\partial v_z}{\partial r} = -\frac{r}{2\mu}\frac{\Delta P}{L} + \frac{A}{r} \qquad\qquad v_z = \frac{r^2}{4\mu}\frac{\Delta P}{L} + Aln(r) + B$$

For oil $0 \leq r \leq R_1$ we have:

$$v_z^o(r) = \frac{-r^2}{4\mu_1}\frac{\Delta P}{L} + Aln(r) + B \qquad\qquad \tau^o(r) = \mu_1\frac{\partial v_z}{\partial r} = -\frac{r}{2}\frac{\Delta P}{L} + \frac{A\mu_1}{r}$$

For water $R_1 \leq r \leq R$ we have:

$$v_z^w(r) = \frac{-r^2}{4\mu_2}\frac{\Delta P}{L} + Cln(r) + D \qquad\qquad \tau^w(r) = \mu_2\frac{\delta v_z}{\delta r} = -\frac{r}{2}\frac{\Delta P}{L} + \frac{C\mu_2}{r}$$

Now the boundary conditions are applied to find the integration constants:

$$\tau^w(0) = 0 \qquad \frac{A\mu_1}{0} \to \infty \qquad \boxed{A = 0}$$

$$\tau^o(R_1) = \tau^w(R_1) \qquad \frac{R_1}{2}\frac{\Delta p}{L} = \frac{R_1}{2}\frac{\Delta p}{L} + \frac{C\mu_2}{R_1} \qquad \boxed{C = 0}$$

$$v_z^w(R) = 0 \qquad \frac{R^2}{4\mu_2}\frac{\Delta p}{L} + D = 0 \qquad \boxed{D = \frac{R^2\Delta P}{4\mu_2 L}}$$

$$v_z^w(R_1) = v_z^o(R_1) \qquad \frac{-R_1^2}{4\mu_1}\frac{\Delta p}{L} + B = \frac{-R_1^2}{4\mu_2}\frac{\Delta p}{L} + \frac{R^2\Delta p}{4\mu_2 L} \qquad \boxed{B = \frac{\Delta P}{4L}\left(\frac{R_1^2}{\mu_1} + \frac{(R^2 - R_1^2)}{\mu_2}\right)}$$

Resulting on the final equations on the form of:

For oil $0 \leq r \leq R_1$

$$v_z^o(r) = \frac{-r^2}{4\mu_1}\frac{\Delta P}{L} + \frac{\Delta P}{4L}\left(\frac{R_1^2}{\mu_1} + \frac{(R^2 - R_1^2)}{\mu_2}\right)$$

$$\boxed{v_z^o(r) = \frac{\Delta P}{4L}\left(\frac{R_1^2 - r^2}{\mu_1} + \frac{R^2 - R_1^2}{\mu_2}\right)}$$

For water $R_1 \leq r \leq R$

$$v_z^w(r) = \frac{-r^2}{4\mu_2}\frac{\Delta P}{L} + \frac{R^2\Delta P}{4\mu_2 L}$$

$$\boxed{v_z^w(r) = \frac{\Delta P}{4\mu_2 L}(R^2 - r^2)}$$

For oil and water $0 \leq r \leq R$

$$\boxed{\tau^o(r) = \tau^w(r) = \frac{r}{2}\frac{\Delta P}{L}}$$

The velocity on the interface is the velocity in R_1 and can be calculated using any of the velocity equations (water or oil) resulting in:

$$v_z(R_1) = \frac{\Delta P}{4\mu_2 L}(R^2 - R_1^2)$$

Other relevant quantities are the shear stress on the wall and the maximum velocity.

$$\tau^w(R) = \frac{R}{2}\frac{\Delta P}{L} \qquad v_z^o(0) = \frac{\Delta P}{4L}\left(\frac{R_1^2}{\mu_1} + \frac{R^2 - R_1^2}{\mu_2}\right)$$

The velocity profile resulting from the equations can be visualized in Figure 1.1.

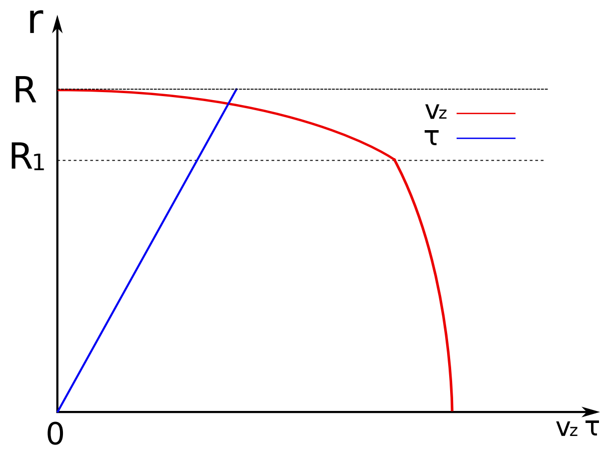

Figure 1.1: Velocity and Shear stress profiles

1.6)

To calculate the flux of each of the fluids it is necessary to integrate the velocity field over the area as following:

For oil:

$$Q_o = \iint_{A_1} v_z^o dA_1 = \int_0^{R_1} \int_0^{2\pi} \left[\frac{-r^2}{4\mu_1} \frac{\Delta p}{L} + \frac{\Delta P}{4L} \left(\frac{R_1^2}{\mu_1} + \frac{(R^2 - R_1^2)}{\mu_2} \right) \right] r dr d\theta$$

$$Q_o = -\frac{r^4 \Delta P}{16\mu_1 L} + \frac{r^2}{2} \frac{\Delta P}{4L} \left(\frac{R_1^2}{\mu_1} + \frac{(R^2 - R_1^2)}{\mu_2} \right) \Bigg|_0^{R_1} 2\pi = -\frac{2\pi R_1^4 \Delta P}{16\mu_1 L} + \frac{R_1^2}{2} \frac{\Delta P}{4L} \left(\frac{R_1^2}{\mu_1} + \frac{(R^2 - R_1^2)}{\mu_2} \right) 2\pi$$

$$\boxed{Q_o = \frac{\pi \Delta p}{4L} \left(\frac{R_1^2}{\mu_2} (R^2 - R_1^2) + \frac{R_1^4}{2\mu_1} \right)}$$

For water:

$$Q_w = \iint_{A_2} v_z^w dA_2 = \int_{R_1}^{R} \int_0^{2\pi} \left[-\frac{r^2 \Delta P}{4\mu_2 L} + \frac{R^2 \Delta P}{4\mu_2 L} \right] r d\theta dr$$

$$Q_w = \left(-\frac{r^4 \Delta P}{16\mu_2 L} + \frac{r^2}{2} \frac{R^2 \Delta P}{4\mu_2 L} \right) \Bigg|_{R_1}^{R_2} 2\pi = \left(-\frac{R^4 - R_1^4 \Delta P}{16\mu_2 L} + \frac{R^2 - R_1^2}{2} \left(\frac{R^2 \Delta P}{4\mu L} \right) \right)$$

$$\boxed{Q_w = \frac{\pi \Delta p}{8\mu_2 L} (R^2 - R_1^2)^2}$$

2 Considering the shock tube shown in Figure 2.1 with initial velocities equal to 0:

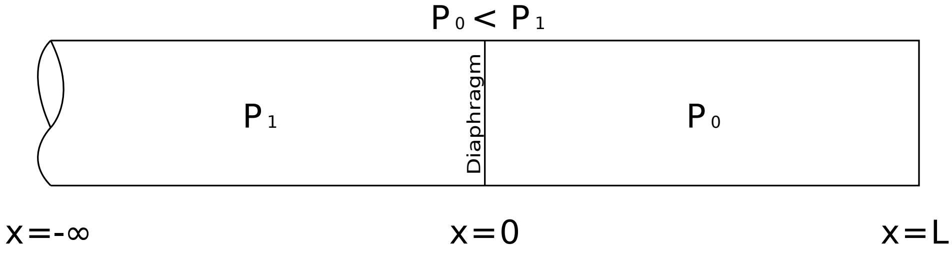

Figure 2.1: Schematic of the problem

The Riemann invariants for the expansion and compression lines may be written as a function of the different pressures:

$$\frac{u}{c} - \frac{1}{\gamma}\frac{p}{p_o} = \text{constant along } x + ct = \text{constant}$$
$$\frac{u}{c} + \frac{1}{\gamma}\frac{p}{p_o} = \text{constant along } x - ct = \text{constant}$$

We know that those expressions remain constant along characteristic lines, therefore by knowing the initial pressure and velocity at both sides of the diaphragm, we can calculate those same quantities at a point after the wave front has passed as shown in Figure 2.2.

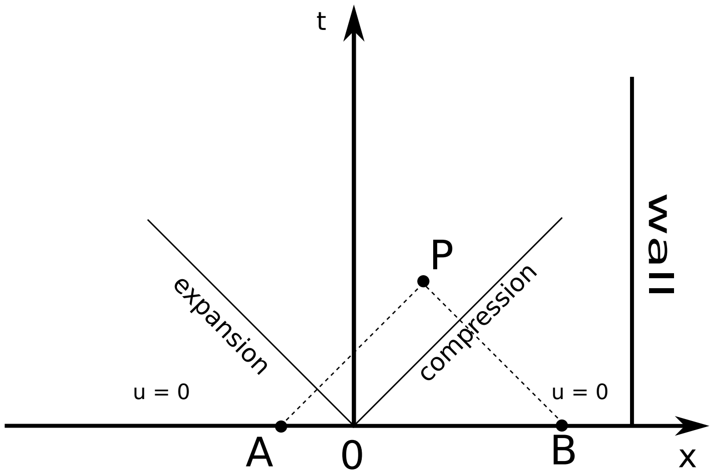

Figure 2.2: Wave front after the diaphragm breaks

From the points P to B $(x + ct)$, knowing that $u_b = 0$ and $p_b = p_o$

$$\frac{u_p}{c} - \frac{1}{\gamma}\frac{p_p}{p_o} = \frac{0}{c} - \frac{1}{\gamma}\frac{p_o}{p_o}$$

$$\frac{u_p}{c} = \frac{1}{\gamma}\left(\frac{p_p}{p_o} - 1\right) \tag{2.1}$$

From the points P to A $(x - ct)$, knowing that $u_a = 0$ and $p_a = p_1$

$$\frac{u_p}{c} + \frac{1}{\gamma}\frac{p_p}{p_o} = \frac{0}{c} + \frac{1}{\gamma}\frac{p_1}{p_o} \tag{2.2}$$

Plugging (2.1) into (2.2)

$$\frac{1}{\gamma}\left(\frac{p_p}{p_o}-1\right)+\frac{1}{\gamma}\frac{p_p}{p_o}=\frac{1}{\gamma}\frac{p_1}{p_o}$$

We get the pressure at point P

$$p_p=\frac{p_o+p_1}{2} \tag{2.3}$$

To find the velocity at that same point we just have to plug (2.3) into (2.1)

$$\frac{u_p}{c}=\frac{1}{\gamma}\left(\frac{p_o+p_1}{2p_o}-1\right)$$

The final expression for velocity at point P is:

$$u_p=\frac{c}{2\gamma}\left(\frac{p_1}{p_o}-1\right) \tag{2.4}$$

To find the pressure and velocity after the wave hits the wall it is ncessary first to find the pressure on the wall as shown in the Figure 2.3, remarking that the $u_w = 0$.

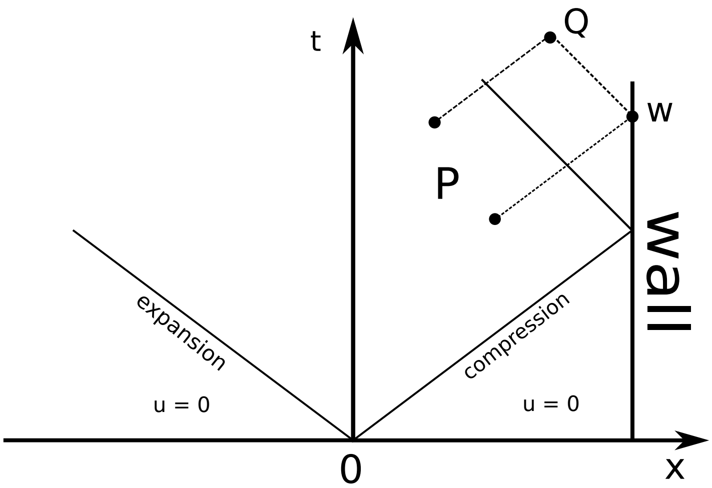

Figure 2.3: Wave front after it hits the wall and characteristic lines

From the points W to P $(x - ct)$

$$\frac{0}{c}+\frac{1}{\gamma}\frac{p_w}{p_o}=\frac{u_p}{c}+\frac{1}{\gamma}\frac{p_p}{p_o} \tag{2.5}$$

We can plug (2.3) into (2.5)

$$\frac{0}{c}+\frac{1}{\gamma}\frac{p_w}{p_o}=\frac{1}{2\gamma}\left(\frac{p_1}{p_o}-1\right)+\frac{1}{\gamma}\frac{p_o+p_1}{2p_o}$$

To get the pressure at the wall

$$p_w=p_1 \tag{2.6}$$

The only remaining pressure and velocities unknown are p_q and u_q, when the wave bounces against the wall and comes back. To find them we use the characteristic lines that pass through points Q-P and Q-W.

From point P to Q $(x - ct)$

$$\frac{u_p}{c} + \frac{1}{\gamma}\frac{p_p}{p_o} = \frac{u_q}{c} + \frac{1}{\gamma}\frac{p_q}{p_o} \tag{2.7}$$

Plugging (2.3) and (2.4) into (2.7) we get:

$$\frac{1}{2\gamma}\left(\frac{p_1}{p_o} - 1\right) + \frac{1}{\gamma}\frac{p_o + p_1}{2p_o} = \frac{u_q}{c} + \frac{1}{\gamma}\frac{p_q}{p_o} \tag{2.8}$$

From point Q to W $(x - ct)$

$$\frac{u_q}{c} - \frac{1}{\gamma}\frac{p_q}{p_o} = \frac{u_w}{c} - \frac{1}{\gamma}\frac{p_w}{p_o}$$

We know that $u_w = 0$ and $p_w = p_1$

$$\frac{u_q}{c} - \frac{1}{\gamma}\frac{p_q}{p_o} = -\frac{1}{\gamma}\frac{p_1}{p_o} \tag{2.9}$$

We have got a linear system of equations formed by (2.8) and (2.9) with p_q and u_q as unknowns. If we solve it we get:

$$p_q = p_1 \qquad\qquad u_q = 0$$

As a summary, we can identify 4 regions in the space-time domain as shown in Figure 2.4 and Table 2.1.

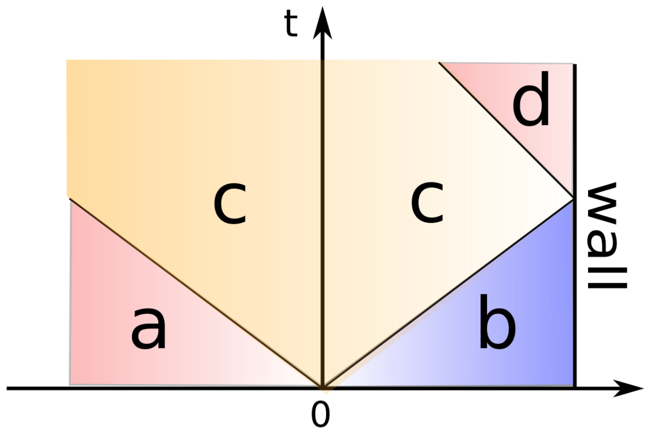

Figure 2.4: Pressure and velocity space-time regions

Table 2.1: Summary

	a	b	c	d
velocity	0	0	$\frac{c}{2\gamma}\left(\frac{P_1}{P_0} - 1\right)$	0
pressure	P_1	P_0	$\frac{P_1+P_0}{2}$	P_1

Looking at the graph, it is clear that for a big enough time the pressure and velocity are going to be the same as region *d* of Figure 2.4.