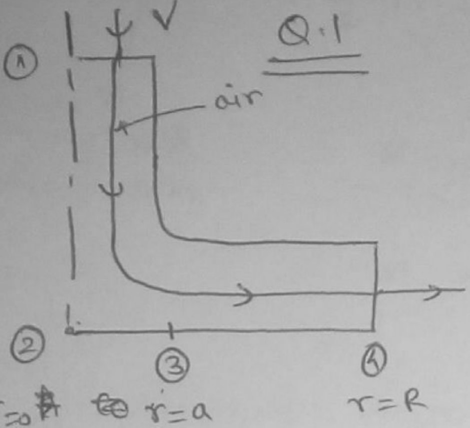
ADVANCED FLUID MECHANICS

12 NOVEMBER 2015

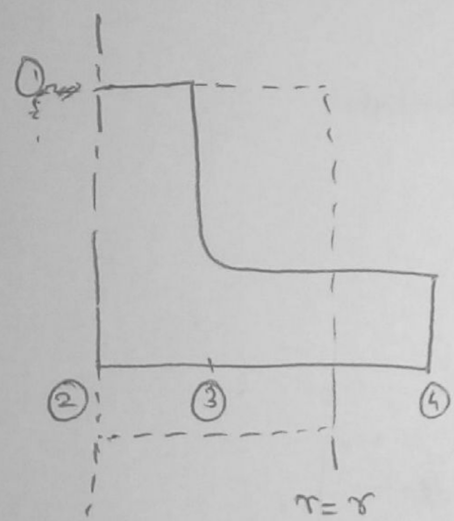
$\boxed{\text{PARIS DILIP MULYE}}$

HOME WORK - 2

Q.1



Taking the following control volume -



Applying Mass conservation in Reynolds Transport theorem form,

$$\frac{DM_{sys}}{Dt} = 0 = \frac{\partial}{\partial t}\int_{CV} \rho \, dv - \int_{CS} \rho \bar{V}\cdot\bar{n} \, ds$$

Assuming steady state,

$$0 = \int_{CS} \rho \bar{V}\cdot\bar{n} \, ds$$

Assuming constant density and using $V_r = f(r)$

$$0 = -\cancel{\rho} V \pi a^2 + 2\cancel{\rho}\pi r h V_r$$

$$\Rightarrow \boxed{V_r = \left(\frac{a^2 V}{2h}\right)\frac{1}{r}}$$

so for $r \geq a$ $\quad V_r = \frac{C_2}{r}$ where $\boxed{C_2 = \frac{a^2 V}{2h}}$

The velocity field should be continous at $r=a$ since there is no mass getting added or removed at that location.

$$\Rightarrow V_r \text{ for } r \leq a \Big|_{r=a} = V_r \text{ for } r \geq a \Big|_{r=a}$$

$$C_1 a = \frac{a^2 V}{2ha}$$

$$\Rightarrow \boxed{C_1 = \frac{V}{2h}}$$

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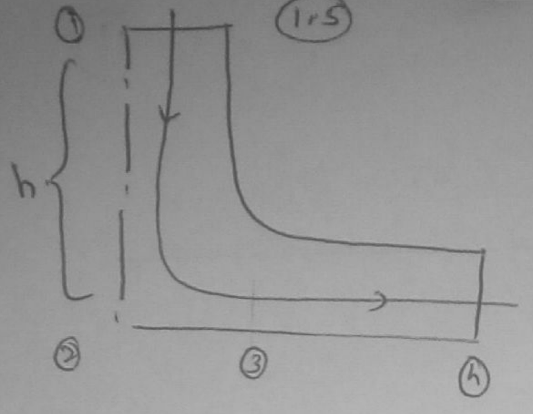
(1.5)



Applying Bernoulli Theorem between 1 & 2

Since the flow is inviscid and assumed to incompressible and with constant ρ.

$$\rho\int_1^2 \underbrace{\frac{\partial \bar{V}}{\partial t}\cdot d\bar{s}}_{0\ \text{(steady state)}} + P_2 + \frac{1}{2}\rho \underbrace{V_2^2}_{\substack{V_2=0 \\ \text{stagnation point}}} - P_1 - \frac{1}{2}\rho V_1^2 - \underbrace{\rho g h}_{\substack{\rho_{air}\ \text{is very less} \\ \text{neglecting term}}} = 0$$

$$\boxed{P_2 = P_1 + \frac{1}{2}\rho V^2}$$

P_1 = is given to be P_0

Applying Bernoulli Theorem between 2 & 3 $\boxed{r \le a}$ (same assumptions as above)

$$P_2 + \frac{1}{2}\rho \underbrace{V_2^2}_{\text{stagnation}} + 0 = P_r + \frac{1}{2}\rho V_r^2 + \underbrace{\rho g h^*}_{\text{small height}}$$

$$P_1 + \frac{1}{2}\rho V^2 = P_r + \frac{1}{2}\rho\left(\frac{Vr}{2h}\right)^2$$

$$P_r - P_1 = \frac{1}{2}\rho V^2\left(1 - \frac{r^2}{4h^2}\right)$$

$$\boxed{P_{r,gauge} = \frac{1}{2}\rho V^2\left(1 - \frac{r^2}{4h^2}\right) \qquad r \le a}$$

Applying Bernoulli Theorem between 2 & 4 $\boxed{r \ge a}$ (same assumptions as above)

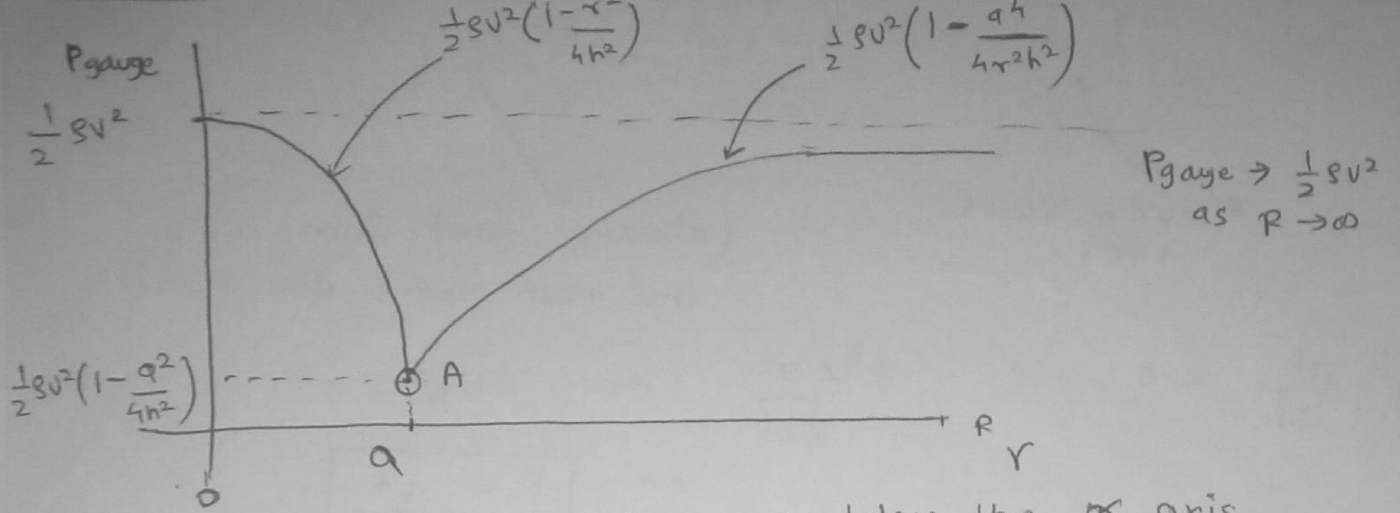
$$P_2 = P_r + \frac{1}{2}\rho V_r^2$$

$$P_1 + \frac{1}{2}\rho V^2 = P_r + \frac{1}{2}\rho\left(\frac{a^2 V}{2rh}\right)^2$$

$$P_r - P_1 = \frac{1}{2}\rho V^2\left(1 - \frac{a^4}{4h^2 r^2}\right)$$

$$\boxed{P_{r,gauge} = \frac{1}{2}\rho V^2\left(1 - \frac{a^4}{4r^2h^2}\right) \qquad r \ge a}$$

Graph



The point A could be above or below the x axis depending ~~on~~ on the value of $1-\frac{a^2}{4h^2}$

if $1-\frac{a^2}{4h^2} < 0$, it goes below x axis, But the overall nature and behavior, shape remains same.

Calculating the Force on cardboard disk.

Total down ward force acting on disk

$$= \int_0^R P_r \, 2\pi r \, dr$$

$$= \int_0^a \frac{1}{2}\rho v^2\left(1-\frac{r^2}{4h^2}\right) 2\pi r\, dr + \int_a^R \frac{1}{2}\rho v^2\left(1-\frac{a^4}{4r^2h^2}\right) 2\pi r\, dr$$

$$= \frac{2\pi}{2}\rho v^2\left[\frac{a^2}{2}-\frac{a^4}{16h^2}\right] + \frac{1}{2}\rho v^2(2\pi)\left[\left(\frac{R^2}{2}-\frac{a^2}{2}\right)-\frac{a^4}{4h^2}\ln\left(\frac{R}{a}\right)\right]$$

Two terms cancel out.

$$\boxed{F_{total}\downarrow = \cancel{2}\pi\rho v^2\left[\frac{R^2}{2}-\frac{a^4}{16h^2}\right] + \pi\rho v^2\left(\frac{a^4}{4h^2}\right)\ln\left(\frac{a}{R}\right)}$$

Q.5

$$F_{down} = \pi \rho v^2 \left[\frac{R^2}{2} - \frac{a^4}{16h^2}\right] + \pi \rho v^2 \left(\frac{a^4}{4h^2}\right) \ln\left(\frac{a}{R}\right)$$

(could be -ve or +ve)

(always -ve) since $a < R$ and rest terms are positive

$$\frac{R^2}{2} - \frac{a^4}{16h^2} < 0 \quad \text{if} \quad R^2 < \frac{a^4}{8h^2} \quad \text{or} \quad 8h^2R^2 < a^4$$

$$\text{or} \quad \boxed{h < \frac{a^2}{2\sqrt{2}R}}$$

so ~~if~~

Fdown could be lees than zero.

Fdown = ± positive or negative term + negative term.

In the given problem

$a = 1E-2$ m

$R = 5E-2$ m

$h = 0.1E-2$ m

$M = 10E-3$ kg

Fdown + weight disk = 0

Fup = weight disk

$$10E-3 \times 9.81 + \frac{\pi}{2}\rho v^2 \left[(5E-2)^2 - \frac{(1E-2)^4}{8(0.1E-2)^2} + \frac{(1E-2)^4}{2(0.1E-2)^2}\ln\left(\frac{1E-2}{5E-2}\right)\right] = 0$$

$\rho_{air} = 1.225$ kg/m3

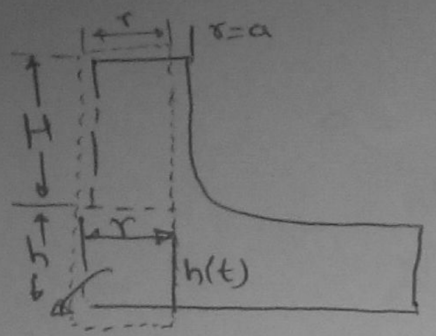
$$10E-3 \times 9.81 = \frac{\pi}{2}\, 1.225\, v^2 \times 6.7972\,E-3$$

$$10E-3 \times 9.81 = 0.013079\, v^2$$

$$v^2 = 7.5$$

$$\boxed{v = 2.7386 \text{ m/s}}$$

$h = h(t)$ No longer a steady state problem.



control volume -

for $r \le a$, Mass continuity in Reynolds Transport Theorem

$$\frac{DM_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{CV} \rho \, dV + \int_{CS} \rho \, \bar{V}\cdot\bar{n} \, ds$$

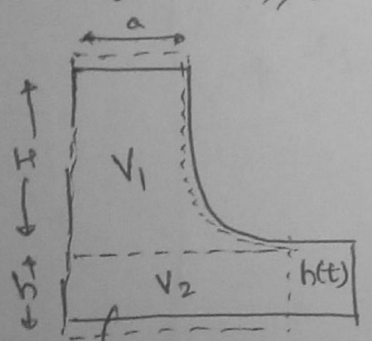
$$0 = \frac{\partial}{\partial t}\left(\rho \pi r^2 (h \cancel{+H})\right) + \rho V_r 2\pi r h - \rho V \pi r^2$$

assuming ρ = constant and H doesnot change.

$$0 = \cancel{\pi} r^2 \frac{dh}{dt} + V_r h 2\cancel{\pi} r - V \cancel{\pi} r^2$$

$$\boxed{V_r = \frac{Vr}{2h} - \frac{r}{2h}\frac{dh}{dt} \qquad r \le a}$$

for $r \geq a$ Mass continuity in Reynolds Transport Theorem and ρ = constant.



control volume

$$\frac{DM_{syst}}{Dt} = \frac{\partial}{\partial t}\int_{CV} \rho \, dV + \int_{CS} \rho \, \bar{V}\cdot\bar{n} \, ds$$

$$0 = \frac{\partial}{\partial t}\left(\cancel{\pi} V_1 + V_2\right) + 2\pi r h V_r - V \pi a^2$$

$$\frac{\partial V_1}{\partial t} = 0 \qquad \frac{\partial V_2}{\partial t} = \pi r^2 \frac{dh}{dt}$$

$$\frac{\pi r^2}{2\pi r h}\frac{dh}{dt} = \frac{V \pi a^2 - 2\pi r h V_r}{2\pi r h}$$

$$\boxed{V_r = \frac{Va^2}{2rh} - \frac{r}{2h}\frac{dh}{dt} \qquad r \geq a}$$

(3.5) Using these expressions and applying Bernoulli between $r=0$ to $r=R$ (2 to 4)

$$\int_0^R \frac{\partial \bar{V}}{\partial t}\cdot d\bar{l} + P_R + \frac{1}{2}\rho V_R^2 - P_2 - \frac{1}{2}\rho \cancel{V_2^2} = 0$$

given $P_R = P_0$

and $P_2 = P_1 + \frac{1}{2}\rho v^2$ and $P_1 = P_0$

$$= P_0 + \frac{1}{2}\rho v^2$$

$$P_R - P_2 = -\frac{1}{2}\rho v^2$$

$$V_R = \frac{Va^2}{2Rh} - \frac{R}{2h}\left(\frac{dh}{dt}\right)$$

$$\int_0^a \frac{\partial \bar{V}}{\partial t}\cdot d\bar{l} + \int_a^R \frac{\partial \bar{v}}{\partial t}\cdot d\bar{l} - \frac{1}{2}\rho v^2 + \frac{1}{2}\rho\left[\frac{Va^2}{2Rh} - \frac{R}{2h}\frac{dh}{dt}\right]^2 = 0$$

$$\boxed{\begin{aligned}&\int_0^a \rho\left[-\frac{Vr}{2h^2}\frac{dh}{dt} + \frac{r}{2h^2}\left(\frac{dh}{dt}\right)^2 - \frac{r}{2h}\frac{d^2h}{dt^2}\right]dr \; + \\ &\int_a^R \rho\left[-\frac{Va^2}{2rh^2}\frac{dh}{dt} + \frac{r}{2h^2}\left(\frac{dh}{dt}\right)^2 - \frac{r}{2h}\frac{d^2h}{dt^2}\right]dr - \frac{1}{2}\rho V^2 \; + \\ &\frac{1}{2}\rho\left[\frac{Va^2}{2Rh} - \frac{R}{2h}\frac{dh}{dt}\right]^2 = 0\end{aligned}}$$

Q.2

(a) lets work in cylindrical polar coordinates. $V = V(r,\theta)$ (2D flow)

BC 1

Irrespective of θ, when $r \to \infty$, $v(r,\theta)$ should become $U\hat{e}_1$

$$\bar{V}(r,\theta)\Big|_{r\to\infty} = U\hat{e}_1$$

BC 2

Velocity along the surface of cylinder should not be radial. ~~It~~ Fluid should move parallel to the boundary. i.e. There cannot be a flux of fluid at this boundary.

$$\bar{V}(r,\theta)\Big|_{r=R} = V_\theta\,\hat{\theta}$$

(b) The flow velocity is $U\hat{e}_1$ so $\nabla\times\bar{V} = 0$. Flow is irrotational initially. Since the flow is inviscid the irrotationality should be conserved. In that case, the stream function should satisfy the Laplace equation.

$$\nabla^2\psi = 0 \quad \text{where} \quad \psi = r^\alpha \sin\theta$$

In polar form, (2D)

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$$

$$0 = \alpha(\alpha-1)r^{\alpha-2}\sin\theta + \frac{1}{r}\times\alpha\times r^{\alpha-1}\sin\theta$$

$$+\frac{1}{r^2}(-r^\alpha)\sin\theta$$

Cancel $r^{\alpha-2}$ from all sides and $\sin\theta$

$$0 = \alpha(\alpha-1) + \alpha - 1$$

$$0 = \alpha^2 - 1$$

$$\boxed{\alpha = \pm 1}$$

(4.5) Since $\nabla^2 \psi = 0$ is a second order diff. Equation. and is linear $\psi_1 + \psi_2 = \psi_3$ is also a solution.

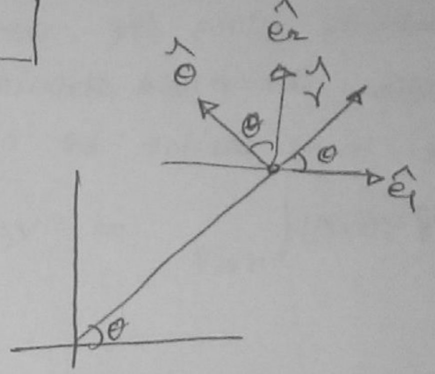
so a general solution can be written as,

$$\psi = \frac{a_1 \sin\theta}{r} + a_2 r \sin\theta$$

(C) ~~so~~

Now a_1, a_2 can be found using BC.

$$\boxed{\begin{aligned} V_r &= \frac{1}{r}\left(\frac{a_1 \cos\theta}{r} + a_2 r \cos\theta\right) \\ V_\theta &= \frac{a_1 \sin\theta}{r^2} - a_2 \sin\theta \end{aligned}}$$



~~as~~ lets check for BC 1

as $r \to \infty$

$V_r \to a_2 \cos\theta \, \hat{r}$

$V_\theta \to -a_2 \sin\theta \, \hat{\theta}$

$\hat{e}_1 = \hat{r} \cos\theta - \hat{\theta} \sin\theta$

$\hat{e}_2 = \hat{r} \sin\theta + \hat{\theta} \cos\theta$

so $V_x = (V_r \cos\theta - V_\theta \sin\theta)\hat{e}_1$

$= a_2 \cos^2\theta + a_2 \sin^2\theta = (a_2)\hat{e}_1$

also $V_y = (V_r \sin\theta + V_\theta \cos\theta)\hat{e}_2$

$= a_2 \cos\theta \sin\theta - a_2 \sin\theta \cos\theta = 0 \, \hat{e}_2$

so as $r \to \infty$
$\bar{V}$ is $a_2 \hat{e}_1$

using BC1 we know as $r \to \infty$ $\bar{V}$ should be $U \hat{e}_1$

$\Rightarrow \boxed{a_2 = U}$

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Now using BC 2

$$\bar{V}(r,\theta)\Big|_{r=R} = \left(\frac{a_1 \cos\theta}{R^2} + a_2\cos\theta\right)\hat{r} +$$

$$\left(\frac{a_1 \sin\theta}{R^2} - a_2\sin\theta\right)\hat{\theta}$$

we get that $V_r = 0$ from BC 2 condition.

$$\frac{a_1\cos\theta}{R^2} + a_2\cos\theta = 0$$

$$a_2 = U$$

$$\frac{a_1 \cancel{\cos\theta}}{R^2} + U = 0$$

$$U = \frac{-a_1}{R^2}$$

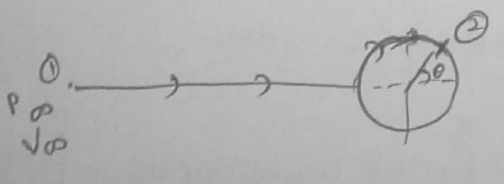
$$\boxed{a_1 = -UR^2}$$

so

$$\cancel{V_r(r) = -U\cos\theta\left(\frac{R^2}{r^2}\right)}$$

$$\boxed{\begin{aligned} V_r &= U\left(1-\frac{R^2}{r^2}\right)\cos\theta \\ V_\theta &= -U\left(1+\frac{R^2}{r^2}\right)\sin\theta \end{aligned}}$$

(d)



Applying Bernoulli theorem from ① to ②

$$P_\infty + \frac{1}{2}\rho V_\infty^2 = P_2 + \frac{1}{2}\rho V_2^2$$

$$P_2 = P_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho V_2^2$$

$\|V_2\| = \|V\|\big|_{r=R} =$ V_r at $r=R \Rightarrow 0$ only V_θ component.

$$V_\theta\big|_{r=R} = \boxed{-2U\sin\theta}$$

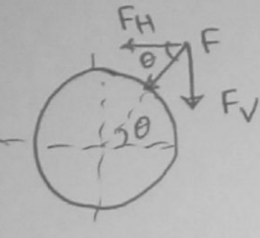
(5.5) $$P_2 = P_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho(2U\sin\theta)^2$$

$$= P_\infty + \frac{1}{2}\rho\left(U^2 - 2U^2\sin^2\theta\right)$$

$$= P_\infty + \frac{1}{2}\rho U^2\left(1 - \sin^2\theta\right)$$

$$\boxed{P_2 = P_\infty + U^2\rho\left(\cos 2\theta - \frac{1}{2}\right)}$$

(e) Net Force on cylinder.



$$F_H = -\int_0^{2\pi} PR\,d\theta\cos\theta$$

$$F_V = -\int_0^{2\pi} PR\,d\theta\sin\theta$$

$$F_H = -\int_0^{2\pi} U^2\rho\left(\cos 2\theta - \frac{1}{2}\right)R\,d\theta\cos\theta = (-U^2\rho)\left[\int_0^{2\pi}\cos 2\theta\,d\theta\cos\theta - \frac{1}{2}\int_0^{2\pi}\cos\theta\,d\theta\right]$$

$$P = \int_0^{2\pi}\cos 2\theta\cos\theta\,d\theta = \cos\theta\int_0^{2\pi}\cos 2\theta\,d\theta - \int_0^{2\pi}\frac{\sin\theta\sin 2\theta}{2}\,d\theta$$

$$= \cos\theta\int_0^{2\pi}\cos 2\theta\,d\theta - \left[\sin\theta\int\sin 2\theta - \frac{P}{2}\right]$$

$$\frac{P}{2} = \left[\frac{-\cos\theta\sin 2\theta}{2}\right]_0^{2\pi} - \left[\frac{\sin\theta\cos 2\theta}{2}\right]_0^{2\pi}$$

$$P = 0$$

$$F_H = -U^2\rho\left[0 - \frac{1}{2}\int_0\cos\theta\,d\theta\right] = 0 \qquad \boxed{F_H = 0}$$

$$F_V = -\int_0^{2\pi} PR\,d\theta\sin\theta = -\int_{-\pi}^{\pi} U^2\rho R\left(\cos 2\theta - \frac{1}{2}\right)\sin\theta\,d\theta$$

$$= -\underbrace{\int_{-\pi}^{\pi} U^2\rho R\cos 2\theta\sin\theta\,d\theta}_{0} + \underbrace{\int_{-\pi}^{\pi}\frac{U^2\rho R}{2}\sin\theta\,d\theta}_{0}$$

(odd function) (odd function)

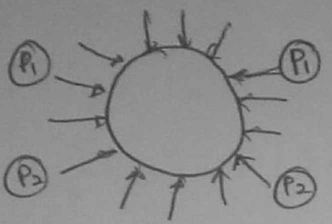
$$\boxed{F_V = 0}$$

Net force is zero. In reality it should have been nonzero value ($F_H \neq 0$) ($F_V = 0$ by symmetry)

But in our problem we are not taking viscosity.

So the fluid cannot apply any force on the cylinder (friction ~~shear~~)

For the force by pressure, the distribution is symmetric about ($\pi - \theta$)



$\cos(2(\pi - \theta)) = \cos(-2\theta) = \cos 2\theta$

So no force ~~in~~ due to pressure as well.

Note:

Description of boundary conditions in terms of ψ.

BC 1

$$V(r,\theta)\Big|_{r\to\infty} = U\hat{e}_1 \Rightarrow V_r = U\cos\theta$$

$$V_\theta = -U\sin\theta$$

$$\Rightarrow \boxed{\begin{aligned} \frac{1}{r}\frac{\partial\psi}{\partial\theta}\Big|_{r\to\infty} &= U\cos\theta \\ -\frac{\partial\psi}{\partial r}\Big|_{r\to\infty} &= -U\sin\theta \end{aligned}}$$

BC 2

$$V_\theta\Big|_{r=R} = -2U\sin\theta \qquad V_r\Big|_{r=R} = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial\psi}{\partial\phi}\Big|_{r=R} = 0 \Rightarrow \boxed{\psi = f(r)}$$

independency on θ.

$$\boxed{-\frac{\partial\psi}{\partial r}\Big|_{r=R} = -2U\sin\theta}$$