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Advanced Fluid Mechanics

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Exercise 1: $\vec{F}$ and $\vec{G}$ vector functions.

a) $\nabla\cdot(\nabla\times\vec{F})=0$

$$\nabla\cdot(\nabla\times\vec{F})=\nabla\cdot\left(P_{ijk}\frac{\partial F_k}{\partial x_j}\vec{e_i}\right)=\frac{\partial}{\partial x_i}\left(P_{ijk}\frac{\partial F_k}{\partial x_j}\vec{e_i}\right)=0 \quad \text{c.q.d.}$$

As an example, for the case of 3D:

$$\nabla\cdot(\nabla\times\vec{F})=\nabla\cdot\left(\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{bmatrix}\right)=$$

$$=\nabla\cdot\left[\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\vec{i}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\vec{j}+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\vec{k}\right]=$$

$$=\cancel{\frac{\partial^2 F_z}{\partial x\partial y}}-\cancel{\frac{\partial^2 F_y}{\partial x\partial z}}+\cancel{\frac{\partial^2 F_x}{\partial y\partial z}}-\cancel{\frac{\partial^2 F_z}{\partial y\partial x}}+\cancel{\frac{\partial^2 F_y}{\partial x\partial z}}-\cancel{\frac{\partial^2 F_x}{\partial y\partial z}}=0$$

b) $\nabla\times(\nabla\times\vec{F})=\nabla(\nabla\cdot\vec{F})-\nabla^2 F$

L.H.S

$$\nabla\times\left(\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{bmatrix}\right)=\nabla\times\left[\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\vec{i}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\vec{j}\right.$$

$$\left.+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\vec{k}\right]=$$

$$= \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} & \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} & \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{bmatrix} =$$

$$= \left(\frac{\partial^2 F_y}{\partial x\,\partial y} - \frac{\partial^2 F_x}{\partial y^2} - \frac{\partial^2 F_x}{\partial z^2} + \frac{\partial^2 F_z}{\partial x\,\partial z}\right)\vec{i} + \left(\frac{\partial^2 F_z}{\partial y\,\partial z} - \frac{\partial F_y}{\partial z^2} - \frac{\partial^2 F_y}{\partial x^2} + \frac{\partial^2 F_x}{\partial x\,\partial y}\right)\vec{j}$$

$$+ \left(\frac{\partial^2 F_x}{\partial x\,\partial z} - \frac{\partial^2 F_z}{\partial x^2} - \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial F_y}{\partial z\,\partial y}\right)\vec{k}$$

R.H.S.

$$\nabla(\nabla\cdot\vec{F}) = \nabla\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) = \left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial y\,\partial x} + \frac{\partial^2 F_z}{\partial x\,\partial z}\right)\vec{i} +$$

$$+ \left(\frac{\partial^2 F_x}{\partial x\,\partial y} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_z}{\partial y\,\partial z}\right)\vec{j} + \left(\frac{\partial^2 F_x}{\partial x\,\partial z} + \frac{\partial^2 F_y}{\partial y\,\partial z} + \frac{\partial^2 F_z}{\partial z^2}\right)\vec{k}$$

$$\nabla^2 \vec{F} = (\nabla^2 F_x, \nabla^2 F_y\ \nabla^2 F_z)$$

$$\nabla^2 F_x = \nabla\cdot(\nabla F_x) = \nabla\cdot\left(\frac{\partial F_x}{\partial x}\vec{i} + \frac{\partial F_x}{\partial y}\vec{j} + \frac{\partial F_x}{\partial z}\vec{k}\right) = \frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2}$$

$$\nabla^2\vec{F} = \left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2}\right)\vec{i} + \left(\frac{\partial^2 F_y}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_y}{\partial z^2}\right)\vec{j} + \left(\frac{\partial^2 F_z}{\partial x^2} + \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2}\right)\vec{k}$$

So,

$$\nabla(\nabla\vec{F}) - \nabla^2\vec{F} = \left(\frac{\partial^2 F_y}{\partial y\,\partial x} + \frac{\partial^2 F_z}{\partial x\,\partial z} - \frac{\partial^2 F_x}{\partial y^2} - \frac{\partial^2 F_x}{\partial z^2}\right)\vec{i} + \left(\frac{\partial^2 F_z}{\partial y\,\partial z} + \frac{\partial F_x}{\partial x\,\partial y} - \frac{\partial^2 F_y}{\partial F x^2} - \frac{\partial^2 F_y}{\partial z^2}\right)\vec{j}$$

$$+ \left(\frac{\partial^2 F_x}{\partial x\,\partial z} + \frac{\partial^2 F_y}{\partial y\,\partial z} - \frac{\partial^2 F_z}{\partial x^2} - \frac{\partial^2 F_z}{\partial y^2}\right)\vec{k}$$ Which is the same as the term in R.H.S.

$$g)\ \nabla\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot\nabla\times\vec{F} - \vec{F}\cdot\nabla\times\vec{G}$$

$$\operatorname{div}(\vec{F}\times\vec{G}) = \varepsilon_{ijk}\frac{\partial(F_j\cdot G_k)}{\partial x_i} = \varepsilon_{ijk}\frac{\partial F_j}{\partial x_i}G_k + \varepsilon_{ijk}\cdot F_j\frac{\partial G_k}{\partial x_i} =$$

$$= G_k\,\varepsilon_{ijk}\frac{\partial F_j}{\partial x_i} + F_j\varepsilon_{ijk}\frac{\partial G_k}{\partial x_i} = \vec{G}\cdot\nabla\times\vec{F} - \vec{F}\cdot\nabla\times\vec{G} \quad \text{c.q.d}$$

For example, for the case 3D.

$$\nabla\cdot(\vec{F}\times\vec{G}) = \nabla\cdot\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ F_x & F_y & F_z\\ G_x & G_y & G_z\end{bmatrix} = \nabla\cdot\Big[(F_yG_z - F_z\cdot G_y)\vec{i}$$

$$+(G_xF_z - F_xG_z)\vec{j} + (F_xG_y - F_yG_x)\vec{k}\Big] = \frac{\partial(F_yG_z)}{\partial x} - \frac{\partial(F_zG_y)}{\partial x}$$

$$+\frac{\partial(G_xF_z)}{\partial y} - \frac{\partial(F_xG_z)}{\partial y} + \frac{\partial(F_xG_y)}{\partial z} - \frac{\partial(F_yG_x)}{\partial z} =$$

$$= \frac{\partial F_y}{\partial x}G_z + F_y\frac{\partial G_z}{\partial x} - F_z\frac{\partial G_y}{\partial x} - G_y\frac{\partial F_z}{\partial x} + G_x\frac{\partial F_z}{\partial y} + F_z\frac{\partial G_x}{\partial y} - \frac{\partial F_x}{\partial y}G_z - F_x\frac{\partial G_z}{\partial y}$$

$$+\frac{\partial F_x}{\partial z}\cdot G_y + F_x\frac{\partial G_y}{\partial z} - \frac{\partial F_y}{\partial z}G_x - F_y\frac{\partial G_x}{\partial z} = G_x\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + G_y\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) +$$

$$+G_z\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) + F_x\left(\frac{\partial G_y}{\partial z} - \frac{\partial G_z}{\partial y}\right) + F_y\left(\frac{\partial G_z}{\partial x} - \frac{\partial G_x}{\partial z}\right) + F_z\left(\frac{\partial G_x}{\partial y} - \frac{\partial G_y}{\partial x}\right)$$

$$= \vec{G}\cdot\nabla\times\vec{F} - \vec{F}\cdot\nabla\times\vec{G} \quad \text{c.q.d.}$$

2. Integral form of 2nd law of Thermodynamics

$$\frac{D}{Dt}\int_{V_t} \rho s \, dV \geq -\int_{S_t} \frac{\vec{q}\cdot\vec{n}}{T} \, dS \qquad s \equiv \text{entropy per unit of mass}$$

- Newtonian fluid $(K \geq 0, \mu > 0)$
- Fourier's law $(\vec{q} = -K\nabla T \quad K > 0)$

$$\int_{S_t} \frac{\vec{q}\cdot\vec{n}}{T} \, dS \underset{\uparrow}{=} \int_{V_t} \nabla\cdot\left(\frac{\vec{q}}{T}\right) dV$$

Gauss Theorem

$$\frac{D}{Dt}\int_{V_t} \rho s \, dV \underset{\uparrow}{=} \int_{V_t} \frac{\partial(\rho s)}{\partial t} dV + \int_{S_t} \rho s \, \vec{v}\cdot\vec{n} \, dS \underset{\uparrow}{=} \int_{V_t} \frac{\partial(\rho s)}{\partial t} dV + \int_{V_t} \nabla\cdot(\rho s \vec{v}) \, dV$$

Reinolds Transport theorem Divergence theorem

$$= \int_{V_t} \left[\frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s \vec{v})\right] dV$$

So, as it has to be accomplished by any Volume of control:

$$\int_{V_t} \left[\frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s \vec{v}) + \nabla\cdot\left(\frac{\vec{q}}{T}\right)\right] dV \geq 0 \quad \forall V_t \Rightarrow$$

$$\operatorname{div}(\phi\vec{A}) = \phi \operatorname{div}\vec{A} + \vec{A} \operatorname{grad}\phi$$

$$\frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s \vec{v}) + \nabla\cdot\left(\frac{\vec{q}}{T}\right) \geq 0; \quad \rho\frac{\partial s}{\partial t} + \rho s \underbrace{\nabla\cdot\vec{v}}_{0} + \rho\vec{v}\,\nabla s + \nabla\cdot\left(\frac{\vec{q}}{T}\right) \geq 0$$

$$\rho\frac{\partial s}{\partial t} + \rho\vec{v}\,\nabla s + \nabla\cdot\left(\frac{\vec{q}}{T}\right) \geq 0$$

Using the relation between energy and entropy:

$$\left.\begin{array}{l} T\,ds = de + p\,d\left(\frac{1}{\rho}\right) \\ \\ \rho \frac{\partial s}{\partial t} + \rho \vec{v}\, \nabla s + \nabla\cdot\left(\frac{\vec{q}}{T}\right) \geq 0 \end{array}\right\}$$

$$\cancel{T}\rho \frac{\partial s}{\partial t} + \rho \vec{v}\, T \nabla s + \nabla\cdot\left(\frac{\vec{q}}{T}\right) T \geq 0;$$

$$\rho \frac{\partial e}{\partial t} + p \frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) + \rho \vec{v}\, \nabla e + p \vec{v}\, \nabla\left(\frac{1}{\rho}\right) + \nabla\cdot\left(\frac{\vec{q}}{T}\right) T \geq 0;$$

Energy conservation

$$\sigma : \nabla \vec{v} - \nabla\cdot\vec{q} + p \vec{v}\, \nabla\left(\frac{1}{\rho}\right) + \nabla\cdot\left(\frac{\vec{q}}{T}\right) T \geq 0$$

For a Newtonian fluid, the constitutive equation is:

$$\sigma = -pI + C : \nabla^s v$$

So that

$$\left(-pI + C : \nabla^s v\right) : \nabla \vec{v} - \nabla\cdot\vec{q} + p \vec{v}\, \nabla\left(\frac{1}{\rho}\right) + \nabla\cdot\left(\frac{\vec{q}}{T}\right) T$$

$$= -p\, \mathrm{div}\, \vec{v} + \lambda\, \mathrm{tr}^2\left(\nabla^s \vec{v}\right) + 2\mu \left(\mathrm{grad}\, \vec{v}\right)^2 - \nabla\cdot\vec{q} + p \vec{v}\, \nabla\left(\frac{1}{\rho}\right) + \nabla\left(\frac{\vec{q}}{T}\right)\cdot T$$