Advanced Fluid Mechanics
Homework 3.

ARTEMII SATTAROV

1a) Dimensionless analysis

	ρ	$\bar{V}_0$	R	R_1	μ_1	μ_2	σ
M	1	0	0	0	1	1	1
L	-3	1	1	1	-1	-1	0
T	0	-1	0	0	-1	-1	-2

Having selected ρ, $\bar{V}_0$, R_1 as primary variables, we define Π-products for remaining variables.

$\Pi_1 = \rho^a \bar{V}_0^b R_1^c \cdot R$

$M^0 L^0 T^0 = (M^1 L^{-3} T^0)^a (L^1 T^{-1})^b (L)^c L^1$

$M^0 L^0 T^0 = M^a L^{1-3a+b+c} T^{-b}$

$a = 0 \quad 1-3a+b+c = 0 \quad -b = 0$

$c = -1 \qquad \Pi_1 = \dfrac{R}{R_1}$

$\Pi_2 = \rho^a \bar{V}_0^b R_1^c \cdot \mu_1$

$M^0 L^0 T^0 = (M^1 L^{-3})^a (L^1 T^{-1})^b L^c (M^1 L^{-1} T^{-1})$

$M^0 L^0 T^0 = M^{1+a} L^{-1-3a+b+c} T^{-b-1}$

$a = -1 \ ; \ b = -1 \ ; \ b+c = -2 \ ; \ c = -1$

$\Pi_2 = \rho^{-1} \bar{V}_0^{-1} R_1^{-1} \mu_1 = \dfrac{\mu_1}{\rho \bar{V}_0 R_1}$

$\Pi_3 = \dfrac{\mu_2}{\rho \bar{V}_0 R_1}$

$\Pi_4 = \rho^a \bar{V}_0^b R_1^c \sigma$

$M^0 L^0 T^0 = (M^1 L^{-3})^a (L^1 T^{-1})^b L^c (M^1 L^0 T^{-2})$

$a+1 = 0 \ ; \ -b-2 = 0 \ ; \ -3a+b+c = 0$

$a = -1 \ ; \ b = -2 \ ; \ c = -1$

$\Pi_4 = \rho^{-1} V_0^{-2} R_1^{-1} \sigma = \frac{\sigma}{\rho V_0^2 R_1}$

Π_4 is a reverse of Weber number ($\Pi_4 = \frac{1}{We}$), which measures surface tension comparing to fluid inertia and is important for waves formation.

When $\sigma \gg \rho V^2 R_1$ waves tend not to form as surface tension stabilizes flow interface

c) In general, gravity is a driving force for waves. However, when densities of two fluid phases are the same, it is reasonable to not consider gravity.

d) Simplification of Navier-Stokes equations:

$$\begin{cases} \rho\left(\cancel{\frac{\partial V_r}{\partial t}} + \cancel{V_r \frac{\partial V_r}{\partial r}} + \cancel{\frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}} - \cancel{\frac{V_\theta^2}{r}} + V_z \cancel{\frac{\partial V_r}{\partial z}}\right) = -\cancel{\frac{\partial p}{\partial r}} + \mu\left(\cancel{\nabla^2 V_r} - \cancel{\frac{V_r}{r^2}} - \cancel{\frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta}}\right) + \rho g_r \\ \quad (V_r = 0) \quad (V_\theta = 0) \quad (V_z \neq f(z)) \quad \substack{p=const \\ \text{for cross sect}} \quad (V_r = 0) \quad (V_\theta = 0) \\ \rho\left(\cancel{\frac{\partial V_\theta}{\partial t}} + V_r \cancel{\frac{\partial V_\theta}{\partial r}} + \cancel{\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta}} + \cancel{\frac{V_r V_\theta}{r}} + V_z \cancel{\frac{\partial V_\theta}{\partial z}}\right) = -\cancel{\frac{1}{r}\frac{\partial p}{\partial \theta}} + \mu\left(\cancel{\nabla^2 V_\theta} - \cancel{\frac{V_\theta}{r^2}} + \cancel{\frac{2}{r^2}\frac{\partial V_r}{\partial \theta}}\right) + \rho g_\theta \\ \quad (V_\theta = 0) \quad \left(\substack{p=const \\ \text{for cross sect.}}\right) \quad (V_\theta = 0) \quad (V_z \neq f(\theta)) \\ \rho\left(\cancel{\frac{\partial V_z}{\partial t}} + V_r \cancel{\frac{\partial V_z}{\partial r}} + \cancel{\frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta}} + V_z \cancel{\frac{\partial V_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \mu \nabla^2 V_z + \rho g_z \end{cases}$$

(no inertial forces in z direct)

(steady) $(V_r = 0)$ $(V_\theta = 0)$ $(V_z \neq f(z))$

$$0 = -\frac{\partial p}{\partial z} + \mu \nabla^2 V_z = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 V_z}{\partial z^2}}$$

$V_z \neq f(\theta)$ $V_z \neq f(z)$

For two layers:

$$\begin{cases} 0 = -\frac{\partial p}{\partial z} + \frac{\mu_1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V_z}{\partial r}\right) & 0 \le r \le R_1 \\ 0 = -\frac{\partial p}{\partial z} + \frac{\mu_2}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V_z}{\partial r}\right) & R_1 \le r \le R \end{cases}$$

Boundary conditions at the interface:

at $r = R_1$ $\tau_{rz\,wat} = \tau_{rz\,oil}$

$V_{z_1}(R_1) = V_{z_2}(R_1)$

Boundary condition at wall for viscous flow:

$V_{z_2}(R) = 0$

e) Solving the equations:

$$-\int \frac{r \Delta P}{\mu_2 L} \partial r = \int \partial\left(r \frac{\partial V_{z_2}}{\partial r}\right)$$

$$-\frac{r^2}{2\mu_2} \frac{\Delta P}{L} + C_1 = r \frac{\partial V_{z_2}}{\partial r}$$

$$\int \left(-\frac{r}{2\mu_2}\frac{\Delta P}{L} + \frac{C_1}{r}\right) \partial r = \int \partial V_{z_2}$$

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$V_{z_2} = -\frac{r^2}{4\mu_2}\frac{\Delta P}{L} + C_1 \ln|r| + C_2$ Using boundary conditions:

$V_{z_2}(R) = 0$; $C_1 \ln|R| + C_2 = \frac{R^2}{4\mu_2}\frac{\Delta P}{L}$

$V_{z_2} = -\frac{r^2}{4\mu_2}\frac{\Delta P}{L} + \frac{R^2}{4\mu_2}\frac{\Delta P}{L} = \frac{(R^2-r^2)}{4\mu_2}\frac{\Delta P}{L}$

In the same way for the inner fluid:

$V_{z_1} = -\frac{r^2}{4\mu_1}\frac{\Delta P}{L} + C_3 \ln|r| + C_4$

at $r = R_1$ $V_{z_1}(R_1) = V_{z_2}(R_1)$; $\frac{(R^2-R_1^2)}{4\mu_2}\frac{\Delta P}{L} = -\frac{R_1^2}{4\mu_1}\frac{\Delta P}{L} + C_3 \ln|r| + C_4$

$C_3 \ln|r| + C_4 = \frac{R^2-R_1^2}{4\mu_2}\frac{\Delta P}{L} + \frac{R_1^2}{4\mu_1}\frac{\Delta P}{L}$

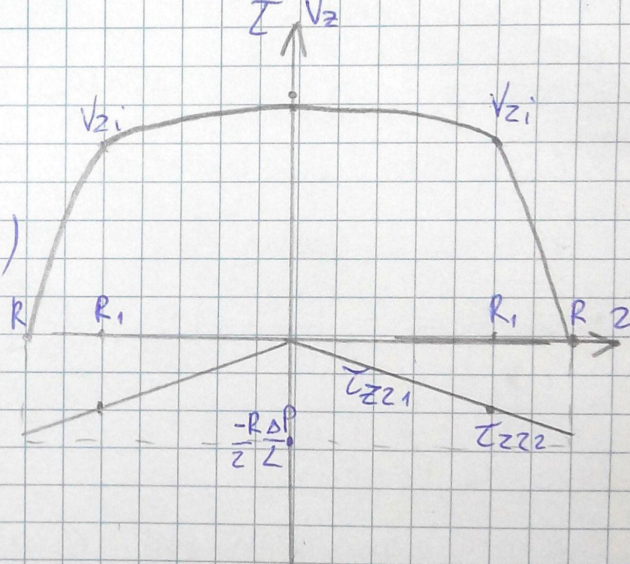
$V_{z_2} = \frac{R_1^2 - r^2}{4\mu_1}\frac{\Delta P}{L} + \frac{R^2-R_1^2}{4\mu_2}\frac{\Delta P}{L}$

$V_{z_i} = \frac{R^2-R_1^2}{4\mu_2}\frac{\Delta P}{L}$ (Interface velocity $(R=R_1)$)

Shear stress

$\tau_{rz_1} = \mu_1 \frac{\partial V_{z_1}}{\partial r} = \mu_1 \frac{-2\cdot r}{4\mu_1}\frac{\Delta P}{L} = -\frac{r}{2}\frac{\Delta P}{L}$

$\tau_{rz_2} = \mu_2 \frac{\partial V_{z_2}}{\partial r} = -\frac{r}{2}\cdot\frac{\Delta P}{L}$



b) Volume flow rates:

$Q_{wat} = \int_{R_1}^{R} 2\pi r V_{z_2}\, dr = 2\pi \int_{R_1}^{R} \left(\frac{r\cdot R^2}{4\mu_2}\frac{\Delta P}{L} - \frac{r^3}{4\mu_2}\frac{\Delta P}{L}\right) dr = \pi\frac{R^4 - R_1^2 R^2}{4\mu_2}\frac{\Delta P}{L} - \pi\frac{R^4 - R_1^4}{8\mu_2}\frac{\Delta P}{L}$

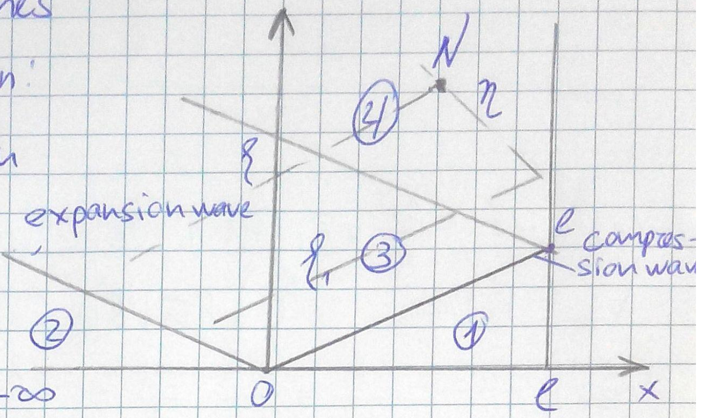
$Q_{oil} = 2\pi \int_0^{R_1} r V_{z_1}\, dr = 2\pi \int_0^{R_1} \left(\frac{r R_1^2}{4\mu_1}\frac{\Delta P}{L} - \frac{r^3}{4\mu_1}\frac{\Delta P}{L} + \frac{r R^2}{4\mu_2}\frac{\Delta P}{L} - \frac{r\cdot R_1^2}{4\mu_2}\frac{\Delta P}{L}\right) dr =$

$= \frac{\pi R_1^4}{4\mu_1}\frac{\Delta P}{L} - \frac{\pi R_1^4}{8}\frac{\Delta P}{L} + \frac{\pi R_1^2 \cdot R^2}{4\mu_2}\frac{\Delta P}{L} - \frac{\pi R_1^2}{4\mu_2}\frac{\Delta P}{L}$.

2

We can illustrate compression wave as well as an expansion wave for small pressure difference as straight lines.

On the diagram the following regions are shown:



1 - region, which is not yet affected by compression wave

2 - region, which is not yet affected by expansion wave

3 - region, which is influenced by waves

4 - region which is also influenced by reflected wave.

In order to find η characteristics we firstly obtain characteristic for 3rd region (ξ_1):

$\frac{u}{c} = \frac{1}{2\gamma}\left(\frac{p_1}{p_0} - 1\right)$; $\frac{p}{p_0} = \frac{1}{2}\left(\frac{p_1}{p_0} + 1\right)$, where p_1 and p_0 are given initial pressures.

η characteristic:

$\frac{u}{c} - \frac{1}{\gamma}\frac{p}{p_0} = -\frac{1}{\gamma}\frac{p_e}{p_0}$, where p_e - pressure on the wall.

Taking into account that $u=0$ at the wall, we obtain: and substituting values from ξ_1:

- $\frac{1}{\gamma}\frac{p_e}{p_0} = \frac{1}{2\gamma}\left(\frac{p_1}{p_0} - 1\right) + \frac{1}{2\gamma}\left(\frac{p_1}{p_0} + 1\right)$
- $\frac{1}{\gamma}\frac{p_e}{p_0} = \frac{1}{\gamma}\frac{p_1}{p}$

Therefore, $p_e = p_1$

Solving equations for η an ξ characteristics for point N:

$$\begin{cases} \frac{u}{c} - \frac{1}{\gamma}\frac{p}{p_0} = -\frac{1}{\gamma}\frac{p_1}{p_0} \\ \frac{u}{c} + \frac{1}{\gamma}\frac{p}{p_0} = \frac{1}{\gamma}\frac{p_1}{p_0} \end{cases} + \qquad \frac{u}{c} = 0 \quad u = 0 \qquad p = p_1$$

The velocity in the point N in the 4th region is zero and pressure is equal to initial pressure to the left of diaphragm p_1.