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Advanced Fluid Mechanics
Homework 4

1) a) Compute the velocity field in cartesian coordinates (u, v) and show that it verifies the boundary conditions. Obtain an expression for the pressure distribution.

$$\psi(r,\theta) = Ur^2 \sin(2\theta)$$

$$\psi(r,\theta) = Ur^2 2 \sin(\theta)\cos(\theta)$$

$x = r\cos\theta$
$y = r\sin\theta$ $\longrightarrow$ $\psi(x,y) = 2Uxy$

$$\boxed{u = \frac{\partial \psi}{\partial y} = 2Ux.} \quad ; \quad \boxed{v = \frac{-\partial \psi}{\partial x} = -2Uy.}$$

then, from the Bernoulli equation, the pressure distribution will be:

$$\boxed{p = p_0 - 2\rho U^2 (x^2 + y^2)}$$

where p_0 is the Bernoulli constant that corresponds to the pressure at the stagnation point.

b) Show that the former velocity and pressure distributions verify the Navier-Stokes equations but not the boundary conditions for the viscous problem.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{-\partial p}{\rho\,\partial x} \rightarrow \boxed{4U^2x = 4U^2x}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \rightarrow \boxed{4U^2y = 4U^2y}$$

It can be observed that the Navier-Stokes equations are verified.

c) A solution for the viscous problem can be obtained modifying the potential flow in such a way that meeting the boundary condition would be posible. If we attempt

$$u = 2Uxf'(y)$$

show that the continuity equation requires that $v = -2Uf(y)$. State appropiate boundary conditions for the function f.

Mass conservation $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\frac{\partial v}{\partial y} = -2Uf'(y) \rightarrow \boxed{v = -2Uf(y)}.$$

with Bcs:

* $f(y) = 0 \rightarrow y = 0$. Impermeability.
* $f'(y) = 0 \rightarrow y = 0$. Non-slip condition.
* $f(y) = y \rightarrow y = \infty$. to Recover the potential flow pressure.

Maria de los Angeles González

d) Use the y-momentum equation to obtain an expression for the pressure distribution in terms of function "f".

$$u\frac{\partial v}{dx} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{dx^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$4U^2 f f' = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2U\mu f''$$

$$2U^2 (f^2)' = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2U\mu f''$$

$$p = \rho(-2U^2 f^2 - 2U\mu f') + h(x).$$

For large values of $y \to F(y) = y \to F'(y) = 1$. thus:

$$P(x,y) = -2\rho \cancel{U^2} y^2 - 2\rho\mu U f' + h(x) = P_0 - 2\rho U^2 (y^2 + x^2)$$

$$h(x) = P_0 - 2\rho U^2 x^2 + 2\rho U\mu$$

$$\boxed{P(x,y) = -2\rho\mu U f' - 2\rho U^2 f^2 + 2\rho U\mu - 2\rho U^2 x + P_0.}$$

e) Using the x-momentum equation and the pressure distribution, obtain a differential equation for the function f. Show that the problem can be solved using the BC stated in point c).

$$\frac{\partial P}{\partial x} = -4\rho U^2 x.$$

x-momentum read as:

$$u\frac{\partial u}{dx} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{\partial x} + \mu\left(\frac{\partial^2 u}{dx^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$4 U^2 x (f')^2 - 4U^2 x f f'' = 4U^2 x + \cancel{\mu} 2Ux f'''$$

$$\boxed{\frac{\mu}{2U} f''' + f f'' - (f')^2 + 1 = 0}$$

It is a third order nonlinear differential equation. three BCs are needed:

$$f(0) = f'(0) = 0 \quad \text{and } f'(y) = 1 \to y = \infty.$$

2) Compare the results with the exact Blasius solution and with the ones obtained assuming a cubic velocity profile.

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2$$

$$\left.\begin{array}{l} u = 0 \quad \text{at } y = 0 \to a = 0. \\ u = U \quad \text{at } y = \delta \to 1 = b + c. \\ \frac{\partial U}{\partial y} = 0 \quad \text{at } y = \delta \to b + 2c = 0 \end{array}\right\} \begin{array}{l} b = 2 \\ c = -1 \end{array}$$

$$\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

María de los Angeles González G.

Computing,

$$\int_0^\delta (U-u)u\,dy =$$

$$\int_0^\delta \left[U-\left(2\left(\frac{y}{\delta}\right)-\left(\frac{y}{\delta}\right)^2\right)U\right]\left(2\left(\frac{y}{\delta}\right)-\left(\frac{y}{\delta}\right)^2\right)U\,dy$$

$$\int_0^\delta U^2\left(2\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)-U^2\left(4\frac{y^2}{\delta^2}-2\frac{y^3}{\delta^3}-2\frac{y^3}{\delta^3}+\frac{y^4}{\delta^4}\right)dy.$$

$$\int_0^\delta U^2\left(2\frac{y}{\delta}-\frac{y^2}{\delta^2}-\frac{4y^2}{\delta^2}+2\frac{y^3}{\delta^3}+2\frac{y^3}{\delta^3}-\frac{y^4}{\delta^4}\right)dy$$

$$\int_0^\delta U^2\left(2\frac{y}{\delta}-5\frac{y^2}{\delta^2}+4\frac{y^3}{\delta^3}-\frac{y^4}{\delta^4}\right)dy=$$

$$U^2\int_0^\delta\left(2\frac{y}{\delta}-\frac{5y^2}{\delta^2}+4\frac{y^3}{\delta^3}-\frac{y^4}{\delta^5}\right)dy=$$

$$=U^2\delta\left(1-\frac{5}{3}+1-\frac{1}{5}\right)=\frac{2}{15}U^2\delta$$

$$\frac{\tau_0}{\rho}=\mu\left(\frac{\partial u}{\partial y}\right)_0=\frac{2U\mu}{\delta}$$

Sustituting in the momentum eq:

$$\frac{2}{15}U^2\frac{\partial\delta}{\partial x}=\frac{2U\mu}{\delta}\rightarrow$$ integrating with $\delta(0)=0$

$$\delta = \sqrt{30}\sqrt{\frac{\mu x}{U}} = 5{,}477\sqrt{\frac{\mu x}{U}}$$

$$\boxed{\frac{\delta}{x} = \frac{5{,}477}{\sqrt{Re}}}$$

Blausius $\frac{\delta}{x} = \frac{5}{\sqrt{Re}}$

Cubic $\frac{\delta}{x} = \frac{4{,}64}{\sqrt{Re}}$

It can be said that the values are in the same order and range.

$$\frac{\theta}{x} = \frac{C_0}{1/2\,\rho U^2} = \frac{4U\mu}{\frac{\delta}{U^2}} = \boxed{\frac{0{,}73032}{\sqrt{Re}}}.$$

Blausius $\frac{\theta}{x} = \frac{0{,}664}{\sqrt{Re}}$.

Cubic $\frac{\theta}{x} = \frac{0{,}646}{\sqrt{Re}}$.